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Chapter 1

Introduction

In this paper, we discuss topics in fixed point theory and specifically focus on the Sarkovskii Theorem. We look at extensions of well-known results for intervals to one-dimensional continuua. In this introduction we give the reader some background into the development of continuum theory and then we will mention the history behind the development of the Sarkovskii Theorem.

The first definition of a continuum was given by Cantor in 1883 [Ing06]. This definition changed over time as new mathematical attributes developed but we see that his definition, although completely different from the definition of a continuum that we use today, is equivalent to the modern continuum definitions. The R.L. Moore definition for continuum, which is the definition that we will give in this paper, is the modern day, commonly used definition of a continuum.

In 1910, Brouwer gave an example of the first indecomposable continuum [Ing06]. This continuum was not actually proven to be an indecomposable continuum until Knaster proved it in 1922. An indecomposable continuum is a continuum that is not decomposable, that is, a continuum that is not the union of two proper subcontinua. In this paper we will work with decomposable continua. Around this time in 1912, Brouwer proved that $n$– cells have the fixed point property [Ing06]. This is a famous theorem that was very useful because it brought up the question of
what other spaces have the fixed point property. In 1916, a proof by Moore brought forth the development of studying a type of continua called chainable continua [Ing06] which are described later in this paper. Chainable continua are a useful type of continua to study because they form naturally as intersections of compact discs in a straightforward manner. Each are the point inverses of continuous $\epsilon$-maps on the interval.

Chainable continua include the arc as one example but also more unique examples. The pseudo-arc is another type of chainable continua. The pseudo-arc is the only arc-like hereditarily decomposable continuum. Bing showed that the pseudo-arc is a common type of continuum in $\mathbb{R}^n$ for $n > 2$ [Nad02].

In 1951, Hamilton proved an important result used in this paper which is that chainable continua have the fixed point property [Ing06]. From there, other types of continua were shown to have the fixed point property. Still today there are unanswered questions about whether or not certain types of continua have the fixed point property.

It is also around this time that Sarkovskii’s Theorem begins to develop. To determine which periodic points imply the existence of others, Coppel proved a result in fixed point theory but Sarkovskii was unaware of the results of Coppel and developed his own ideas which extended Coppel’s ideas [BH11]. Sakovskii published these results in the 1960s but his work was not known to anyone outside of Eastern Europe until after 1975 [BH11]. These published results are what is known today as the famous Sarkovskii Theorem.

In 1975, Li and Yorke presented a theorem that proves a special case of Sarkovskii’s Theorem from the 1960s. Li and Yorke had not known of Sarkovskii’s Theorem at
this time. Their work extended Sarkovskii’s by showing that periodic points of period 3 not only imply points of all other periods but also results in a phenomena called chaos. This was the first use of the word in the field.

It was after this that Yorke was approached at a conference by Sarkovskii who informed Yorke by using non-verbal communication (they did not speak the same native language) that he had already proven the theorem that Li and Yorke presented along with a more extensive result [BH11]. The popularity of Li and Yorke’s publication lead to the popularity of Sarkovskii’s Theorem.

As Sarkovskii’s Theorem became popular, many mathematicians began proving Sarkovskii’s Theorem using different methods. The proof presented for Sarkovskii’s Theorem in this text is not Sarkovskii’s original proof but one that was given by Block, Guckenheimer, Misiurewicz, Young, Burkhart, Ho, Morris and Straffin which is the standard proof of the theorem. This complete proof can be found in [Dev03]. In 1989, Minc and Transue showed that Sarkovskii’s Theorem which deals with the space of the reals, can be applied to a Hereditarily Decomposable Chainable Continuum.

This paper is organized as follows. In chapter 2 we give some basic definitions and examples related to continuum theory and fixed point theory. In chapter 3 and 4 we discuss the standard fixed point theorems of Brower and Hamilton. Specifically, we compare the techniques used in the proof of those theorems. In chapter 5, we give sarkovskii’s theorem and sketch the proof of sarkovskii’s theorem. In chapter 6, we sketch the argument that Minc and Transue give for extending Sarkovskii’s theorem to decomposable chainable continua. Lastly, in chapter 7, we compare the proofs of the theorem for the reals and for continua. From here we see the obstacles
to attempting to apply Sarkovskii’s theorem to further extensions.
Chapter 2

Definitions

This chapter discusses the basic definitions and gives some illustrative examples of the ideas used throughout the paper.

We begin with some basic definitions in fixed point theory.

**Definition** A *fixed point* of a function $f$ is a point $p$ such that $f(p) = p$.

**Definition** A space $X$ has the *fixed point property* provided that every continuous mapping from $X$ to $X$ has a fixed point.

**Definition** Given a continuous function $f$ from $X$ to $X$, a point $x \in X$ is called a *periodic point of $f$* if there exists a positive integer $n$ such that $f^n(x) = x$ where $f^n$ is the $n$th iterate of $f$.

The positive integer $n$ satisfying $f^n(x) = x$ is called the *period* of the point $x$.

If $n$ is the least positive integer satisfying $f^n(x) = x$ for $x$ then $x$ has *prime period* $n$.

We will now give some definitions of terms in continuum theory. These definitions can vary by text but we will give the most standard definitions.

**Definition** A *continuum* is a nonempty, compact, connected metric space.

We will give many examples of this later in the paper but we give the reader one simple example.
Example 2.1 An interval $I = [1, 7] \in \mathbb{R}$.

Definition A subcontinuum is a continuum that is a subset of a given space.

Example 2.2 Continuing Example 2.1: a subcontinuum of $I$ is the interval $[3, 5] \in \mathbb{R}$.

Definition A space $X$ is called nondegenerate if it consists of more than one point.

Definition Let $(X, d)$ be a continuum. A chain in $X$ is a nonempty, finite indexed collection $C = \{U_1, U_2, ..., U_n\}$ of open subsets $U_i$ of $X$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Definition Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a continuous mapping. Then $f$ is called an $\epsilon$-map if the diameter of $f^{-1}(f(x)) < \epsilon$ for all $x \in X$.

Later, we give example 2.7.

Theorem 2.3 A continuum $X$ is chainable if and only if for each $\epsilon > 0$, there is an $\epsilon$-map $g$ of $X$ onto $[0, 1]$.

This is a common theorem in continuum theory so we omit it’s proof.

Definition A decomposable continuum is a continuum that can be written as the union of two proper subcontinua.

Example 2.4 Using $I$ from example 2.1 we see that $I$ is a decomposable continuum because it can be written as $[1, 5] \cup [5, 7]$ where $[1, 5]$ and $[5, 7]$ are both continua.

Definition A hereditarily decomposable continuum is a continuum whose nondegenerate subcontinua is decomposable. That is, a continuum is hereditarily decomposable if every subcontinua itself is decomposable.
Definition An indecomposable continuum is a continuum that is not decomposable.

Definition Let \( \{ X_i, f_i \}_{i=1}^{\infty} \) be an inverse sequence of coordinate spaces \( X_i \), and continuous functions \( f_i : X_{i+1} \rightarrow X_i \). Then the inverse limit of \( \{ X_i, f_i \}_{i=1}^{\infty} \) is the subspace of the cartesian product space \( \prod_{i=1}^{\infty} X_i \) defined by

\[
\text{lim}_{\rightarrow} \{ X_i, f_i \}_{i=1}^{\infty} = \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \right\}
\]

Theorem 2.5 If each \( X_i \) is a continuum the the inverse limit of \( \{ X_i, f_i \}_{i=1}^{\infty} \) is also a continuum.

Example 2.6 The Knaster (Buckethandle) Continuum. For each \( i = 1, 2, \ldots \), let \( X_i = [0, 1] \) and let \( f_i : X_{i+1} \rightarrow X_i \) be defined by

\[
f_i(x) = \begin{cases} 
2x : & 0 \leq x \leq 1/2 \\
-2x + 2 : & 1/2 \leq x \leq 1 
\end{cases}
\]

then the inverse limit \( \text{lim}_{\rightarrow} \{ X_i, f_i \}_{i=1}^{\infty} \) is an indecomposable continuum. This continuum is due to Knaster who was the first to discover an indecomposable continuum. See figure 2.6 below.
Example 2.7  Let $C$ be a chain such that $C = \{U_1, U_2, ..., U_n\}$. Let $a$ and $b$ be distinct points in $\mathbb{R}^2$. Then there are simple chains $C_n$ in $\mathbb{R}^2$ for $n = 1, 2, ...$, such that $\{U_1, U_2, ..., U_n\}$ have diameter less than $1/2^n$ and satisfy

1. For each $n = 1, 2, ..., C$ goes from $a$ to $c$ through $b, C_{3n+2}$ goes from $b$ to $c$ through $a$, and $C_{3n+3}$ goes from $a$ to $b$ through $c$ and

2. For each $n = 1, 2, ..., \cup C_n \supset \cup C_{n+1}$. Then $X \in \mathbb{R}^2$ such that $X = \cap_{n=1}^\infty (\cup C_n)$ is a continuum that is indecomposable. See figure 2.7 on the next page.

![Diagram](image.png)

**Definition**  A hereditarily indecomposable continuum is a continuum whose nodegenerate subcontinua is indecomposable.

**Definition**  A space is said to be irreducible between the points $a$ and $b$ provided that it is connected and these two points cannot be joined by any closed connected set
which is different from the whole space. In this case, the point $a$ is said to be a point of irreducibility of the space.

Throughout this paper, when we refer to a continuum that is irreducible about its endpoints $a$ and $b$ we use the notation $<a, b>$ where $a$ and $b$ are points in the continuum, $<a, b)$ where $b$ is not included in the continuum and $(a, b>$ where $a$ is not included in the continuum.

We now give some examples of continua that are irreducible about certain points.

**Example 2.8** Continuing example 2.1: the interval $I$ is irreducible about its endpoints 1 and 7. There is no proper subcontinuum of $I$ that contains both 1 and 7.

It can actually be shown that any interval is irreducible about its endpoints. While the interval is a type of continuum that is irreducible about its endpoints we will use the standard $[a, b]$ notation for an interval to differentiate between an interval and an unspecific continuum.

**Example 2.9** The Topologist’s Sine Curve defined by the closure of $T$ for

$$T = \{(x, \sin(1/x)) : \mathbb{R}^2 : 0 < x \leq 1\}$$

is a continuum that is irreducible between the points $(-1, \sin(-1))$ and $(1, \sin(1))$. See figure below.
**Definition** Consider the function $g$. Let $X$ be a metric space. Given a mapping $g : X \to Y$, we call *point inverses* of $g$ the inverse images of single points of $Y$, in other words $g^{-1}(y)$ for $y \in Y$.

The point inverses $g^{-1}(t)$ of this function $g$ will be called the *layers* $T$ of the space $X$ irreducible between $a$ and $b$.

**Definition** Let $X \subseteq X$ be a continuum with end layers $I_0$ and $I_1$. We will say that $I$ is *central* with respect to the orbit of $x$ if the follow conditions are satisfied:

1. there are points $x_i \in I_0$ and $x_j \in I$ such that $x_i < x_j$, $f(x_j)$ is either in $I_0$ or to the left of it and $f(x_i)$ is either in $I_1$ or to the right of it

2. $I - (I_0 \cup I_1)$ does not contain points of the orbit of $x$ and

3. $I$ is a maximal continuum satisfying (1) and (2), and $I$ is not contained in one layer of $X$.

We are now ready to delve into the topics of fixed point theory and continuum theory.
Chapter 3

Some Fixed Point Theorems

In this section we will discuss some basic fixed point theorems with proofs. We
will also discuss a commonly used proof technique which is called the dog-chases
rabbit argument. This argument which we describe later in the paper is an informal
technique that is used to prove given spaces have the fixed point property.

Theorem 3.1 The unit interval \( I \) has the fixed point property.

Proof Assume \( I \to I \) is a continuous mapping. Define

\[
A = \{x \in I : x \leq f(x)\}, \text{ and} \\
B = \{x \in I : x \geq f(x)\}
\]

Since all points \( x \in I \) are either mapped to the left, right or same spot as \( x \), we have
that \( I = A \cup B \). Since \( 0 \in A \) and \( 1 \in B \) and by definition of \( A \) and \( B \), we have that \( A \)
and \( B \) are nonempty closed sets. Since \( I \) is connected and \( I = A \cup B \), then we may
choose a \( y \in A \cap B \). Thus

\[
y \leq f(x) \tag{3.1}
\]

\[
y \geq f(x). \tag{3.2}
\]

Hence by (1) and (2) \( f(y) = y \). Therefore, \( I \) contains a fixed point.
Another way we could have proven that the unit interval has the fixed point property would be to use a method made popular by R.H. Bing which can be found in [Dev03] which is called the *dog chases rabbit argument*.

Imagine a road which is the space $X$ we are considering. The variable $x$ represents the position of dog and $f(x)$ represents the position of the rabbit. The dog chases the rabbit and when the position of the rabbit and the dog are the same (the dog catches the rabbit), $x = f(x)$ and we have a fixed point for our space.

For the unit interval, imagine that the dog starts out a position $x=0$. The dog will chase the rabbit to the point $x=1$ which is where the road stops. The rabbit cannot escape the road and since the road eventually hits a dead end, the rabbit must either stop at the dead end of the road or pass by the dog. In either case, the dog will catch the rabbit. Thus the unit interval has a fixed point.

For the next argument we use the following definition:

**Definition** An arc is a space that is homeomorphic to the closed interval $[0,1]$.

**Theorem 3.2** An arc has the fixed point property.

**Proof** Let $X$ be an arc. Let $h : I$ be the unit interval $[0,1]$. Let $h : I \rightarrow X$ be a *homeomorphism* (a continuous bijective mapping with continuous inverse function). Recall from above that $I$ has the fixed point property. Let $f : X \rightarrow X$ be a continuous mapping. We will show that $f$ has a fixed point. Consider $h^{-1} \circ f \circ h : I \rightarrow I$. Since $I$ has the fixed point property and $h^{-1} \circ f \circ h : I \rightarrow I$, we may choose an $x \in I$ such that

$$(h^{-1} \circ f \circ h)(x) = x.$$
Thus

\[(h^{-1}(f(h(x)))) = x \quad \text{if and only if} \quad f(h(x)) = h(x).\]

Therefore \(f\) has a fixed point and the arc \(X\) has the fixed point property.

Notice that the only attribute of the unit interval that we used in this proof is the fact that it has the fixed point property. For this reason this proof can be generalized to show that any space which is homeomorphic to a space with the fixed point property has the fixed point property as well.

For showing that the arc has the fixed point property we could have easily used the dog chases rabbit argument in a similar fashion to that of the unit interval since an arc is homeomorphic to the unit interval.

In example, 2.9 we defined the Topologist’s Sine Curve and we will now show that this space also has the fixed point property.

**Theorem 3.3** The Topologist’s Sine Curve has the fixed point property.

Instead of giving a formal proof of this theorem we will use the previously mentioned dog chases rabbit argument.

**Proof** Let let \(p\) denote the position of the dog and let \(f(p)\) denote the position of the rabbit. The dog starts out at \((1, \sin(1))\) and follows along the \(\sin(1/x)\) arc going from \(p\) to \(f(p)\). However, this path on the road never ends and the rabbit always remains in front of the dog.

Noticing that path of the road the dog is on is never ending, the dog switches positions to the point \((0, 1)\). The rabbit is then still at the point \(f(0,1)\) or \(f(p)\).
Thus the rabbit will be at the point \((0, y)\) for some \(-1 \leq y \leq 1\). This means the dog will eventually reach the rabbit by traveling south. Therefore, we see that the Topologist’s Sine Curve has the fixed point property.
Chapter 4

Chainable Continua have The Fixed Point Property

In this section we will give some a standard proof due to Hamilton that Chainable Continua have the fixed point property. The proof of the theorem in Chapter 6 relies heavily on this result. We follow Hamilton’s proof as it is given in [Nad05].

**Theorem 4.1** All Chainable Continua have the fixed point property.

**Proof** Assume to the contrary that $X$ is a chainable continua for which there exists a continuous map $f$ from $X$ to $X$ with no fixed point. Consider $A = \{d(x, f(x)) : x \in X\}$ which is the continuous image of a compact set, so it is compact. Since

$$d(x, f(x)) = 0 \quad \text{if and only if} \quad f(x) = x$$

which contradicts that $f$ has no fixed point,

$A$ is a set of positive integers that contains it’s boundary points. Let $\varepsilon = \inf(A)$. Since $\varepsilon \in A$, $\varepsilon > 0$.

Let $C = \{U_1, U_2, ..., U_n\}$ be an $\varepsilon/2$ - chain covering $X$.

Let $V = \{x \in X :\text{ if } x \in U_i \text{ and } f(x) \in U_{i'}, \text{ then } i < j\}$.

Let $W = \{x \in X :\text{ if } x \in U_i \text{ and } f(x) \in U_{i'}, \text{ then } i > j\}$.

We will now show that $V$ and $W$ are nonempty open sets in $X$. 

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Let \( v \in V \). Let \( U_i \) and \( U_j \) be links of \( C \) such that \( v \in U_i \) and \( f(v) \in U_j \). Then \( i < j \).

Since \( f \) is continuous, there is an open subset \( G \in U_i \) such that \( v \in G \) and \( f(G) \subset U_j \).

We will now show \( G \subset V \). Let \( x \in G \). We will show that \( x \in V \). Let \( U_k \) and \( U_l \) be links of \( C \) such that \( x \in U_k \) and \( f(x) \in U_l \). If \( k < l \) then \( x \in V \).

We now show \( k < l \). Since \( x \in G \subset U_i \) and \( f(x) \in f(G) \subset U_j \), then \( U_i \cap U_k \neq \emptyset \) and \( U_j \cap U_l \neq \emptyset \). Thus

\[
k \in \{i - 1, i, i + 1\} \text{ and } l \in \{j - 1, j, j + 1\}
\]

Since \( C \) is \( \epsilon/2 \)-chain and \( d(x, f(x)) \geq \epsilon \) for all \( x \in X \), since \( x \in U_k \) and \( f(x) \in U_l \), if \( U_k \cap U_l \neq \emptyset \) then \( d(x, f(x)) < \epsilon \). This contradicts \( d(x, f(x)) \geq \epsilon \).

Thus

\[
|k - l| \geq 2
\]

and \( k \neq l \).

Assume to the contrary that \( k \geq l \). By (2) we have

\[
2 \leq k - l \quad \text{by (1)}
\]

\[
\leq (i + 1) - (j - 1)
\]

\[
= i - j + 2
\]

So \( 2 \leq i - j + 2 \). Thus \( j \leq i \). This contradicts our choice of \( U_i \) and \( U_j \). Thus \( k < l \).

Hence \( x \in V \). Thus \( G \subset V \). Therefore, \( V \) is open in \( X \). Similarly, \( W \) is open in \( X \).

We will now show \( X = V \cup W \). Assume to the contrary that \( p \in X \setminus (V \cup W) \). Then we may choose links \( U_m, U_r, U_s \) and \( U_t \in C \) such that \( p \in U_m \cap U_r, f(p) \in U_s \cap U_t \), \( s < m \) and \( r < t \). Since \( U_m \cap U_r \neq \emptyset \), \( |m - r| \leq 1 \) and \( r = \{m - 1, m, m + 1\} \). Since \( r < t \), \( m \leq t \). Hence \( s < m \) we have \( s < m \leq t \). Thus, since \( U_s \cap U_t \neq \emptyset \), we must have
\[ t = s + 1 \] and hence, that \( m = s + 1 \). Thus \( p \in U_{s+1} \). So, \( f(p) \in U_s \), \( d(p, f(p)) < \epsilon \). This contradicts our choice of \( \epsilon \). Therefore \( X = V \cup W \).

Since \( X = V \cup W \) and \( X \) is connected, we may choose \( q \in V \cap W \). Thus there are links \( U_a, U_b, U_c \) and \( U_d \in C \) such that \( q \in U_a \cap U_b \), \( f(q) \in U_c \cap U_d \), \( a < c, d < b \). Since \( d < b \) and \( U_a \cap U_b \neq \emptyset \) and by how we defined \( \epsilon \) we can use the triangle inequality to get the following:

\[
    d \leq \min\{a, b\} - 2 \quad \text{and} \quad \max\{a, b\} + 2 \leq c.
\]

Thus \( c - d \geq 4 \). However, since \( U_c \cap U_d \neq \emptyset \), \( c - d \leq 1 \). This is a contradiction. Therefore \( f : X \to X \) has a fixed point.
Chapter 5

Sarkovskii’s Theorem and Proof for Intervals

Before we give Sarkovskii’s Theorem, we will first mention the Sarkovskii’s ordering of the natural numbers.

**Definition** Consider the following ordering of the natural numbers:

\[
3 \succ 5 \succ 7 \succ 9 \succ \ldots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \ldots \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \succ \ldots
\]

\[
\succ 2^3 \cdot 3 \succ 2^3 \cdot 5 \succ 2^3 \cdot 7 \succ \ldots \succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \ldots \succ 2^2 \succ 2 \succ 1
\]

This ordering of the natural numbers is called *Sarkovskii’s Ordering*.

We now state Sarkovskii’s famous result, now commonly known as the Sarkovskii Theorem.

**Theorem 5.1** (*Sarkovskii’s Theorem*) Assume \( f \) is a map from \( \mathbb{R} \) to \( \mathbb{R} \). Suppose \( f \) has a periodic point of prime period \( m \). If \( m \prec n \) in the Sarkovskii ordering, then \( f \) also has a periodic point of prime period \( n \).

For this proof, we will only show it is true for one case, that of odd \( n \). We follow the basic proof given by Block, Guckenheimer, Misiurewicz, Young, Burkhart, Ho, Morris and Straffin given in [Dev03] which also contains the missing cases of the proof.
Proof Assume $X$ has a periodic point $x$ of period $n$ for an odd natural number and $n > 1$. Assume $f$ has no periodic points of odd period less than $n$. Let $x_1, x_2, ..., x_n$ be the points on the orbit of $x$, enumerated from left to right. Then $f(x_n) < x_n$ and $x_1 < f(x_1)$. We may choose the largest $i$ for which $f(x_i) > x_i$. Let $I_1$ be the interval $[x_i, x_{i+1}]$. Since $f(x_{i+1}) < x_{i+1}$ and $f(x_i) < x_i$.

Since $f(x_{i+1}) = x_i$ and $f(x_i) = x_{i+1}$ implies a period of length 2 then $f(I_1)$ contains at least one interval of the form: $[x_j, x_{j+1}]$. Let $O_2$ be the union of all intervals of the form $[x_j, x_{j+1}]$ that are covered by $f(I_1)$. Thus $I_1 \subset O_2$ with $I_1 \neq O_2$. Let $I_2$ be any interval in $O_2$ of the form $[x_j, x_{j+1}]$, then $I_1 \rightarrow I_2$ (or $f(I_1)$ covers $I_2$).

Let $O_3$ denote the union of intervals of the form $[x_j, x_{j+1}]$ that have the property that they are covered by the image of some interval in $O_2$.

Let $O_{l+1}$ be the union of intervals that are covered by the image of some interval $O_l$. If $I_{l+1}$ is any interval in $O_{l+1}$, then there is a collection of $I_2, I_3, ..., I_l$ with $I_j \subset O_j$ which are such that $I_1 \rightarrow I_2 \rightarrow ... \rightarrow I_l \rightarrow I_{l+1}$.

So $O_l$ forms a union of intervals that increase in size as $l$ increases. Since there are finitely many $x_j$, we may choose an $l$ such that $O_l = O_{l+1}$. Also note, for this choosen $l$, $O_l$ contains all intervals of the form $[x_j, x_{j+1}]$. To see why this is true, suppose there are intervals of the form $[x_j, x_{j+1}]$ that are not covered by $O_l$. Then we could create a period of $f$ with period less than $n$. This would contradict that $n$ is the least period of $n$ for $n$ odd.

Since $n$ is odd, then there are an odd number of points $x_j$ for the orbit of $x$. Since $I_1$ is an interval between two points that are successive in the orbit of $x$, then there remain an odd number of $x_j$s that lie to the left and right of $I_1$. Thus at least one point must be mapped to a point on the opposite side of $I_1$. So there is at least one
interval whose image must cover $I_1$. This implies that for some $O_k$ whose image covers $I_1$, there is an interval $[x_j, x_{j+1}]$ in $O_k$ which is not equal to $I_1$.

Thus we have $I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow ... \rightarrow I_k \rightarrow I_1$. Let us choose the smallest $k$ for which we have this path. Then this is the shortest indirect path from $I_1$ into $I_1$. Note that by definition each $I_i$ is of the form $[x_j, x_{j+1}]$ for some $j$ and $I_1 \neq I_2$. Thus for some $k$ we have the following mapping:

Since intervals have the fixed point property, we see that this path leads to a periodic point of period length $k + 1$. Suppose $k + 1 < n$. Since $n$ is odd and $n$ is the smallest odd period length then for $k$ which is less than $n - 1$ then it can be shown that $k = n - 1$. Since we chose $k$ to be the smallest integer with this path we see that it can now be shown that we have periods for all $m$ where $m < n$.

We now have periods larger than $n$ since we have the mappings of $f$ from $I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow ...I_{n-1} \rightarrow I_1 \rightarrow ...I_1$. We also have periods of even length for even
integers smaller than \( n \) by following the path \( I_{n-1} \rightarrow I_{n-2} \rightarrow I_{n-1} \) which shows \( f \) has a period 2 point, by following the path \( I_{n-1} \rightarrow I_{n-4} \rightarrow I_{n-2} \rightarrow I_{n-4} \rightarrow I_{n-1} \) which shows \( f \) has a period 4 point and so on.

This is as much of the proof as we will give and the rest can be found in the source. We have shown this part of the proof to use as a reference in a later section.

For other cases such as \( n = 2^k \cdot m \), where \( m \) is odd, the above argument applies to \( g = f^{2^k} \). The remaining case of \( n = 2^k \) is fairly straightforward.
Chapter 6

Sarkovskii's Theorem for Hereditarily Decomposable Chainable Continua

In this section we will state and prove 2 cases of Sarkovskii’s Theorem for hereditarily decomposable chainable continua. We will follow the basic proof as presented by Minc and Transue found in [MT89] and will give some added details along the way.

It can be shown that a continuum $X$ is chainable if and only if there exists an $\epsilon$-map from $X$ onto the unit interval $[0, 1]$. We define this mapping to be $g$ and we say that for $a$ and $b \in X$ if $g(a) < g(b)$ then we will call $a$ less than $b$ or $a < b$. We may also say that $a$ is left of $b$ in $X$. Thus if $a < b < c$ then $b \in < a, c>$.

Before we can prove this theorem we must first give some preliminary lemmas that will be used in the proof of the theorem. Many of the proofs of these lemmas are omitted but we included some of the proofs to add to the discussion presented in Section 7. All proofs for all of these lemmas may be found in [MT89].

We list these lemmas by their order of appearance in the part of the proof.

**Lemma 6.1** (Lemma 4.3 in [MT89]) There is a continuum $I$ central with respect to the orbit of $x$.

**Proof** Let $I \subset X$ be a continuum with end layers $I_0$ and $I_1$. Choose a natural number $i$ such that $x_i$ is the greatest number in the orbit of $x$ with $x_i < f(x_i)$. Since
\( f(x_{i+1}) < x_{i+1} \) and \( x_i < x_{i+1} \), then we have that

\[
f(x_{i+1}) \leq x_i < x_{i+1} \leq f(x_i).
\]

Then \( x_i \) is mapped to a point in the layer of \( x_{i+1} \) or to the right of it and \( x_{i+1} \) is mapped in the layer of \( x_i \) or to the left of it. Thus we see that \( I = \langle x_i, x_{i+1} \rangle \) is central with respect to the orbit of \( X \).

**Lemma 6.2** (Lemma 4.5 in [MT89]) Let \( I \) be a continuum central with respect to the orbit of \( x \). Let \( I_0 \) be the left end layer of \( I \) and let \( I_1 \) be a right end layer. Suppose that

- there is a point of \( I_0 \) which is sent by \( f \) to \( I_0 \) or to the left of it or

- there is a point of \( I_1 \) which is sent by \( f \) to \( I_1 \) or to the right of it.

Then \( X \) contains periodic points of all periods.

**Lemma 6.3** (Lemma 4.4 in [MT89]) Let \( I \) be a continuum irreducible between two points of the orbit of \( x \) such that \( I \subset f(I) \). Let \( z \) be a point of the orbit of \( x \) such that \( I \) is contained in the layer \( z \) in \( < z, f(z) \rangle \). Then \( I \) is not central with respect to the orbit of \( x \).

**Lemma 6.4** (Lemma 2.4 in [MT89]) Let \( X \) be a hereditarily decomposable chainable continuum. Let \( a, b, c \) and \( d \) be points of \( X \). Let \( A \) and \( B \) denote the layers of \( < a, b \rangle \) containing \( a \) and \( b \), respectively. Let \( C \) and \( D \) denote the layers of \( < c, d \rangle \) containing \( c \) and \( d \), respectively. Suppose that \( f \) is a mapping of the irreducible continuum containing \( a, b, c \) and \( d \) into \( X \) such that \( f(a) = c, f(b) = d, f(d) \in A \) and \( b \in (a, f(c)) \). Assume also that either \( < a, b \rangle \cap < c, d \rangle \subset B - C \) or \( < a, b \rangle \cap < c, d \rangle \subset B - D \). Then \( f \) has a periodic point of period 2.
Lemma 6.5 (Lemma 4.6 in [MT89]) Let $I = < x_i, x_j >$ be a continuum central with respect to the orbit of $x$. Let $J \subset X$ be a continuum central and let $k \geq 1$ be an integer such that

1. $J = < y, z >$, $y$ and $z$ are points of the orbit of $x$, $y, z$ and $f(z)$ are on the same side of $< x_i, x_j >$, but $f(y)$ is on the other side and
2. $J \subset f^k(I)$

Then $X$ contains a periodic point of odd period $m$ with $1 < m \leq k + 2$.

Lemma 6.6 (Lemma 4.1 in [MT89]) If a continuum $I \subset X$ contains at least two points of the orbit of $x$ and $I \subset f(I)$, then $X \subset f^{n-2}(I)$.

Lemma 6.7 (Lemma 4.12 in [MT89]) If a continuum $I \subset X$ contains at least two points of the orbit of $x$, $I \subset f(I)$ and $X \subset f^{n-3}(I)$ then we may represent the orbit of $x$ such that

1. $y_1 = f(y_n)$ and $y_{j+1} = f(y_j)$ for $j = 1, 2, ..., n - 1$, and
2. $y_1, y_2 \in I$ and $y_{j+2} \in f^j(I)$ for $j = 1, 2, ..., n - 2$.

Lemma 6.8 (Lemma 2.6 in [MT89]) Let $X$ be a hereditarily decomposable chainable continuum. Let $a, b, c$ and $d$ be points of $X$. Let $A$ and $B$ denote the layers of $< a, b >$ containing $a$ and $b$, respectively. Let $C$ and $D$ denote the layers of $< c, d >$ containing $c$ and $d$, respectively. Suppose that $f$ is a mapping of the irreducible continuum containing $a, b, c$ and $d$ into $X$ such that $f(a) = c$, $f(b) = d$, $f(d) \in A$ and $b \in (a, f(c))$. Assume also that either $< a, b > \cap < c, d > \subset B - C$ or $< a, b > \cap < c, d > \subset B - D$. Then $f$ has a periodic point of period 2.

Recall from theorem 4.1 that chainable continuua have the fixed point property. We will use the following notation:
which shows that $f^n$, for any natural number $n$, is a recursive mapping from either a subset of $A$ onto itself or a subset $B$ onto itself. Thus if $A$ and $B$ are continuums, then there is a periodic point of period $n$ for $A$ and $B$. If $A, B \subseteq X$ then $X$ contain a periodic point of period $n$.

**Theorem 6.9** Let $Y$ be a hereditarily decomposable chainable continuum and let $X$ be a subcontinuum of $Y$. If $n > m$, $f : X \rightarrow Y$ is a continuous map and $f$ has a periodic point of period $n$, then $f$ has a point of period $m$.

**Proof** Case 1. Assume $n = 3$. We wish to show $f$ has periodic points of all other periods. By Lemma 6.1, we may choose an $I = [x_i, x_j]$ that is central with respect to the orbit of $x$.

We have three choices for our choice of $I$. We begin with letting $I = [x_1, x_3]$. Since $I$ is central with respect to the orbit $x$, $f(x_1) \subseteq I_0$ or $f(x_1)$ is to the left of $I_0$. Thus by Lemma 6.2, $f$ contains points periodic points of all periods.

We now consider $I = [x_1, x_2]$ or $I = [x_2, x_3]$. Without loss of generality, let $I = [x_2, x_3]$. Since $I$ is central with respect to the orbit of $x$, $x_2 < f(x_2)$. Since $f(x_2) \neq x_1$, $f(x_2) = x_3$.

Since $x_1$, $x_2$ and $x_3$ are points in the orbit of $x$ for $n=3$, $f^2(x_2) \neq x_2$. If $f(x_3) = x_2$, then $f^2(x_2) = f(f(x_2)) = f(x_3) = x_2$ which gives $f$ a periodic point of period 2. We thus move forward with $f(x_3) \neq x_2$, so $f(x_3) = x_1$ and $f(x_1) = x_2$.

Assume $x_1 \in [x_2, x_3]$. Since $f(x_1) = x_2$, then $f^{-1}(x_2) = x_1$ and $x_1$ is in a layer of $x_2$ in $[x_2, x_3]$. Thus by Lemma 6.2, $f$ contains periodic points of all other periods.
Remark We cannot have \( x_3 \notin < x_1, x_2 > \). To see this assume to the contrary that
\( x_3 \in < x_1, x_2 > \). Since \( f^{-1}(x_3) = x_2 \), then \( x_2 \) is contained in the layer of \( x_3 \) in
\( < x_3, f(x_3) >= < x_3, x_1 > \). Thus by Lemma 6.3, \( I \) is not central with respect to the orbit of \( x \). This contradicts that \( I \) is central with respect to the orbit of \( x \). So we proceed by assuming that \( x_3 \notin < x_1, x_2 > \).

Let \( L \) denote the layer of \( x_2 \) in the \( < x_1, x_3 > \). Since neither continua \( < x_1, x_2 > \) and \( < x_2, x_3 > \) contains the other, \( L \) is not an end layer of \( < x_1, x_3 > \).

Let \( W = \text{cl}(< x_1, x_2 > -L) \) and let \( Z = \text{cl}(< x_2, x_3 > -L) \). Since \( L \) is nowhere dense, \( < x_1, x_3 >= W \cup Z \).

We now have two cases where \( x_2 \) could be.

Case 1.1. Let \( x_2 \in W \). Then \( W = < x_1, x_2 > \). Also, \( < x_1, x_2 > \cap < x_2, x_3 > = \emptyset \). We have that

\[
f^2(x_1) = f(f(x_1)) \\
= f(x_2) \\
= x_3
\]

and

\[
f^2(x_2) = f(f(x_2)) \\
= f(x_3) \\
= x_1
\]

So \( x_2 \in < x_1, x_3 >= (f^2(x_2), f^2(x_1)) \). Thus by Lemma 6.4 there exists a \( t \in < x_1, x_2 > \) such that \( f^2(t) = x_2 \). Since \( t \in < x_1, x_2 > \) then \( < x_1, t > \cap < x_2, x_3 > = \emptyset \). We have:
Thus we have periodic points of all periods greater than 2.

We will now show that $f$ has a periodic point of period 2. We have $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1 \in < x_1, x_2 >$ and $x_2 \in (x_1, x_3) = (x_1, f(x_2))$. Let $B$ be the layer of $x_2$ in $< x_1, x_2 >$ and let $D$ be the layer of $x_3$ in $< x_2, x_3 >$.

Since $< x_1, x_2 > \cap < x_2, x_3 > \subseteq B - D$, then $f$ has a periodic point of period 2.

Case 1.2 Let $x_2 \in Z$. Then $Z = < x_2, x_3 >$ and $(x_2, x_3) = \emptyset$.

We will first prove $f$ has a periodic point of period 2. Let $b \in [x_2, x_3]$ such that $f(b) = x_2$. Let $a = x_3$, $c = x_1$ and $d = x_2$. By Lemma 6.8, we see that $f$ has a periodic point of period 2.

We will now show that $f$ has a periodic point of period 4. Let $x$ be a periodic point of period 3 for $f^2$. We have shown that $f$ having a periodic point of period 3 implies $f$ has a periodic point of period 2. Thus since $f^2$ has a periodic point of period 3, by similar argument, $f^2$ has a periodic point of period 2. Thus $f$ also has a periodic point of period 4.

Let $I_1$ be the layer of $x_3$ in $< x_2, x_3 >$. If $f(x_3) \in I_1$ then by Lemma 6.2, $f$ has periodic points of all periods.

We proceed by assuming $f(x_3) \notin I_1$. Since $x_2 \in < f(x_3), f(x_2) >= < x_1, x_3 >$, by Lemma 6.4, we may choose a $q \in < x_2, x_3 >$ such that $f(q) = x_2$. Let $Q$ be the layer of $q$ in $< x_2, q >$.

If $f(x_2) = x_3 \in f(Q)$, then $Q \subseteq f(Q)$ and $< x_1, x_2 > \supseteq f^2(Q)$. Since $< x_1, x_2 >$ and $Q$ are disjoint then:
Thus \( f \) has periodic points of all periods greater than 2.

We now proceed by assuming \( x_3 \notin f(Q) \). Since \( q \in \langle x_3, x_2 \rangle = \langle f(x_2), f(q) \rangle \), by Lemma 6.4 we may choose a \( t \in \langle x_2, q \rangle \) such that \( f(t) = q \). We have \( \langle t, q \rangle \subseteq f(\langle t, q \rangle) \)

\[
\begin{align*}
f^2(t) &= f^2(f(t)) \\
&= f^2(q) \\
&= f(f(q)) \\
&= f(x_2) \\
&= x_3
\end{align*}
\]

and

\[
\begin{align*}
f^3(q) &= f^2(f(q)) \\
&= f^2(x_2) \\
&= f(f(x_2)) \\
&= f(x_3) \\
&= x_1
\end{align*}
\]

So we have that \( \langle x_1, x_2 \rangle \subseteq f^3(\langle t, q \rangle) \). Let \( m \) be any integer greater than 4. We have:

\[
\begin{align*}
\langle x_1, x_2 \rangle & \xrightarrow{f} Q \\
\langle x_1, x_2 \rangle & \xrightarrow{f^m} Q
\end{align*}
\]

Then we see that we may choose a point \( z \in \langle x_1, x_2 \rangle \) such that \( f^m(z) = z \) and
\( f^k(z) \in < t, q > \) for \( 1 \leq k \leq m - 3 \). Since \( < x_1, x_2 > \cap < t, q > \neq \emptyset \), we can assume that \( f^k(z) \neq z \) for \( 1 \leq k \leq m - 1 \).

Thus \( z \) is a periodic point of period \( m \). To see this, assume to the contrary that \( f^{m-2}(z) = z \) or \( f^{m-1}(z) = z \).

If \( f^{m-1}(z) = z \), then we have

\[
    f(z) = f(f^{m-1}(z)) \\
    = z
\]

This gives us that \( m = 1 \) which is a contradiction. If \( f^{m-2}(z) = z \), then we have

\[
    f^2(z) = f^2(f^{m-2}(z)) \\
    = z
\]

Thus \( m - 1 = 2 \) or \( m - 2 = 2 \). In either case, \( m \) is equal to a value less than 4 which contradicts our definition of \( m \). Therefore we have periodic points for all values greater than 4 and the proof for this case is complete.

We now prove one more case of the theorem.

Case 2: Let \( n \) be an odd number greater than 3. Let \( I = < x_i, x_j > \) be a continuum central with respect to the orbit of \( x \) and let \( I_0 \) and \( I_1 \) be the left and right end layers of \( I \). Let \( m \) be the number of points on the same side of \( < x_i, x_j > \) as \( x_i \). Since \( n \) is an odd number, then \( m \neq n - m \). Thus we may choose a point \( x_s \) such that \( x_s \) and \( f(x_s) \) are on the same side of \( < x_i, x_j > \). Without loss of generality, assume that \( x_s \) is on the same side of \( < x_i, x_j > \) as \( x_i \). Let \( J = < x_s, x_i > \). Since \( x_s \), \( x_i \) and \( f(x_s) \) are on the same side of \( < x_i, x_j > \), \( f(x_i) \) is on the other side of \( < x_i, x_j > \) (Recall that \( x_i \) is in \( I_0 \) with \( I \) central with respect to the orbit of \( x \) and \( J = < x_s, x_i > \subset f^n(I) \), then by
Lemma 6.5 X, contains a periodic point of odd period m with $1 < m \leq n + 2$. Since $I \subset J$ and $I$ contains $x_i$ and $x_i$ and $I \subset f(I)$, Lemma 6.6 implies that $J \subset f^{n-2}(I)$. Since $n$ was chosen as the smallest nontrivial odd period of $f$, then $f$ does not contain a periodic point of odd period less than or equal to $n - 1$. Thus by Lemma 6.5, $J \subset f^{n-3}(I)$. By Lemma 6.7, we may choose $y_1, y_2 \in I$ and $y_{j+2} \in f^j(I) - f^{j-1}(I)$ for $j = 1, 2, ..., n - 2$. In can be shown in [MT89] that

1. $y_n < y_{n-2} < ... < y_3 < y_1 < y_2 < y_4 < ... < y_{n-3} < y_{n-1}$.

   and

2. $[y_{n-4}, y_{n-1}]$ and $[y_n, y_{n-2}]$ are disjoint.

We will now construct a periodic point of period 2. Let $a = y_n, b = y_{n-2}, c = y_1$ and $d = y_{n-1}$. Let $B$ be the layer of $y_{n-2}$ in $< y_n, y_{n-2}>$. Let $D$ denote the layer of $y_{n-1}$ in $< y_1, y_{n-1}>$. Thus by (2) above we have $< y_n, y_{n-2} > \cap < y_1, y_{n-1} > = < a, b > \cap < c, d > \subset B$. Thus we have by Lemma 6.8 that $f$ has a periodic point of period 2.

We will now construct a periodic point of period 4. We have shown that $x$ is a periodic point of period $n$ for $f^2$. Thus using the previous construction we can show that $f^2$ has a periodic point of period 2. Obviously, the point has period 4 under $f$.

It can be shown in [MT89] that it is then possible to construct periodic points for periods greater than or equal to $n - 1$ but we omit the details as the construction is similar but more complicated to the construction given in the previous case.
Chapter 7

Elements of Sarkovskii’s Theorem for a Continuum versus the Reals

In this section we will discuss the similarities and differences between Sarkovskii’s Theorem when applied to an interval in the Reals versus when it is applied to a continuum. We will refer to the parts of the proofs that we have described in the sections 5 and 6.

7.1 Outline of Proof

We begin by outlining the proof given in section 5 for the interval for the case when $f$ was given to have a period of odd length $n$ for some $n > 1$.

For the interval, we first numbered the points on the orbit of $x$ for a periodic point of $f$ called $x$ where $x$ has least period $n$. We then defined an interval, $I_1$ with left endpoint $x_i$ such that $x_i$ is the greatest number in the orbit of $x$ for which $x_i < f(x_i)$. The right endpoint of this interval $I_1$ is the successive point in the orbit of $x$, named $x_{i+1}$.

Now clearly, $x_{i+1}$ is mapped to a point on the left side of the interval $I_1$. From there we show that each point to the left of $I_1$ must be mapped to the right of the $I_1$ and each point to the right of $I_1$ must be mapped to a point on the left of $I_1$. This brings other defined intervals $I_2, I_3, ...$ that lie on the left (or right) of $I_1$ for even
indicies \( i \) of \( I \), and on the right (or left) of \( I \) for odd indices \( i \).

These intervals then only intersect at points in the orbit of \( x \). From there we construct an iteration of \( n \) mappings of \( f \) and show that for any \( m \) where \( n \succ m \), the \( m \) iteration of these mappings go from an interval \( I \) onto itself. Since each of these intervals have the fixed point property, we see that the mapping \( f^m \) has a fixed point and so we have a periodic point of period \( m \).

7.2 Ordering Points in Chainable Continua

The proof of Sarkovskii’s Theorem for the interval depended on the fact that we have an ordering for points in an interval. A continuum does not necessarily have an ordering for points in its space. Instead, we need to define a different type of ordering for our continuum.

To create a type of ordering for our space, we use the fact that we are considering only chainable continua. Thus we can define an \( \epsilon \)-map \( g \) from our continuum \( X \) onto the unit interval \([0, 1]\). We then have a sort of ordering for the points in our space and we can order the point inverses of \( g \) in \( X \) by where they lie on the unit interval.

7.3 Reducing the Space

Another difference arises because of the fact that we do not necessarily have intervals for the space between the points that are in the orbit of \( x \). Thus we must use the notation that we are considering spaces that are irreducible between the points
in the orbit of \( x \). For example, we use \( I_i = \langle x_i, x_{i+1} \rangle \) to represent the space of points that are irreducible between the points \( x_i \) and \( x_{i+1} \) in the orbit of \( x \).

If \( g \) is a bijective function, then we see that, similar to the proof in section 5, that the only overlap of the \( I_i \) is at points in the orbit of \( x \).

However \( g \) is not necessarily a bijective function so we then use the idea of layers of points to discuss where these intervals overlap and to show that we still have \( m \) iterations of mappings of \( f \) from a space \( I_i \) onto itself for any \( m \) where \( n \succ m \).

We also had to use the fact that we have a hereditarily decomposable continuum \( X \) for which we are proving the Sarkovskii result. If we could not decompose our continuum into \( X \) into unions of continua then we would not be able to use the same proof technique.

### 7.4 Central Set

Both of these theorems begin with the same basic idea. We consider an interval \( I_1 \) or an irreducible space \( I_1 \). To know that this type of space exists in the continuum \( X \), we use the fact from Lemma 6.1 that every continuum with a mapping \( f \) such that \( f \) has a periodic point \( x \) with orbit \( x \) contains a continuum \( I \) where \( I \) is central with respect to the orbit of \( x \).

When comparing the proof of Lemma 6.1 we see that we use the same set up for a continuum \( I \) (or \( I_1 \) in the proof) as what is developed at the begining of the proof in section 5. To prove we have a continuum \( I \) which is central with respect to the orbit of \( x \), we choose a natural number \( i \) such that \( x_i \) is the greatest number in the orbit of \( x \) with \( x_i < f(x_i) \). Then we have that \( f(x_{i+1}) \leq x_i < x_{i+1} \leq f(x_i) \). Then
we see that $I = \langle x_i, x_{i+1} \rangle$ is central (meaning $x_i$ is mapped to the left of $I$ and $x_{i+1}$ is mapped to the right of $I$. This almost the same as the begining of the theorem in section 5.

### 7.5 Fixed Point Property

Both of these theorems also depend on an important fact of fixed point theory which is that in the proof of either cases we are dealing with spaces that have the fixed point property. If our spaces $I_i$ and $I_i$ did not have the fixed point property then our proofs would not be valid.

This was just a discussion of the outlines of the techniques used in the proofs of both of these theorems and the actual proofs involve a lot more details. It is not as easy to verify what we have claimed to show in this section so we encourage the reader to examine the details of these proofs in [Nad05] and [MT89].
Bibliography


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