Sampling Probability

For his 10th birthday party, Daniel’s parents have decided to include a new activity. Daniel and his friends will take turns selecting three Jolly Ranchers from a bag containing a total of 100 Jolly Ranchers. The distribution of the flavors is given in the table below. The first child to select at least one Lemon-flavored Jolly Rancher will win a new Easton RVY 1150 Rival baseball glove. The rules of the game are as follows:

Game Rules:

1. Each player is granted only one turn.
2. Each player can only draw three Jolly Ranchers from the bag. If more than three Jolly Ranchers are drawn on a given turn, the player’s turn is forfeited.
3. After each turn, the selected Jolly Ranchers must be returned to the bag.
4. To win the new glove, a player must select at least one Lemon-flavored Jolly Rancher.

Distribution of Flavors:

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Watermelon</td>
<td>24</td>
</tr>
<tr>
<td>Apple</td>
<td>24</td>
</tr>
<tr>
<td>Cherry</td>
<td>24</td>
</tr>
<tr>
<td>Blue Raspberry</td>
<td>12</td>
</tr>
<tr>
<td>Grape</td>
<td>12</td>
</tr>
<tr>
<td>Lemon</td>
<td>4</td>
</tr>
</tbody>
</table>

Questions:

1. How many different ways can a player select three Jolly Ranchers from the population of 100?
2. What are the possible winning selections?
3. What is the probability that a player selects exactly one Lemon-flavored Jolly Rancher?
4. What is the probability that a player selects no Lemon-flavored Jolly Ranchers?
5. What is the probability that a player selects at least one Lemon-flavored Jolly Rancher?
Answers:

1. Since the order in which the Jolly Ranchers are selected makes no difference, we can use combinations to answer this question. Since we are selecting three objects from a set of 100, we have:

   \[ \binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{100!}{3!97!} = 161,700 \]

2. In order to win the new baseball glove, a player needs to select at least one Lemon-flavored Jolly Rancher. Since the phrase “at least one” is equivalent to “one or more,” we know a player will win if their selection looks like any of the following:

   Exactly one Lemon-flavored Jolly Rancher is selected:
   
   - Non-Lemon, Non-Lemon, Lemon
   - Non-Lemon, Lemon, Non-Lemon
   - Lemon, Non-Lemon, Non-Lemon

   Exactly two Lemon-flavored Jolly Ranchers are selected:
   
   - Non-Lemon, Lemon, Lemon
   - Lemon, Non-Lemon, Lemon
   - Lemon, Lemon, Non-Lemon

   All three Jolly Ranchers are Lemon-flavored:
   
   - Lemon, Lemon, Lemon

3. We want the probability that a player selects **exactly one** Lemon-flavored Jolly Rancher. We have several ways to approach this problem. For now, we can stick with combinations. We already know there are 161,700 ways to select three Jolly Ranchers from the population of 100. The challenge is to determine how many of those 161,700 selections have exactly one Lemon-flavored Jolly Rancher. Let’s consider a few cases before we start crunching numbers!

   - Watermelon, Apple, Lemon
   - Watermelon, Cherry, Lemon
   - Apple, Lemon, Cherry
   - Blue Raspberry, Grape, Lemon

Since the order in which the Jolly Ranchers are selected makes no difference, we could treat the above case as:

   - Lemon, Watermelon, Apple
   - Lemon, Watermelon, Cherry
   - Lemon, Apple, Cherry
   - Lemon, Blue Raspberry, Grape
Looking at it this way, we can take advantage of the Fundamental Counting Principle. We will multiply the number of ways we can select one Lemon Jolly Rancher from the four Lemon Jolly Ranchers in the population by the number of ways we can select two non-Lemon Jolly Ranchers from the 96 non-Lemon Jolly Ranchers in the population:

\[
\binom{4}{1} \binom{96}{2} = \frac{4!}{1!3!} \times \frac{96!}{2!94!} = 4 \times 4560 = 18,240
\]

So, our final answer is:

\[
\frac{\binom{4}{1} \binom{96}{2}}{100 \binom{3}{3}} = \frac{18,240}{161,700} = 0.112801484 \approx 0.113
\]

In other words, a player has about an 11.3% chance of selecting exactly one Lemon-flavored Jolly Rancher on their turn.

As an alternative, we could solve this problem using what we know about “AND” probabilities. The three selected Jolly Ranchers are essentially picked in succession without replacement, meaning we have three dependent events. We have to be a little more careful if we use this approach. If we use:

\[
P(A \text{ and } B \text{ and } C) = P(A) \times P(B \mid A) \times P(C \mid (A \text{ and } B)),
\]

the assumption is that events $A$, $B$, and $C$ occurred in that order – event $A$ first, then event $B$, and finally event $C$.

Let’s start by assuming the Lemon-flavored Jolly Rancher is picked last. That leaves us with $P(\text{non-Lemon and non-Lemon and Lemon})$. Following the formula above, we need to break this down into three parts. The first part of the product represents the probability of selecting a non-Lemon flavored Jolly Rancher from the population of 100. Since there are 96 Jolly Ranchers that are not Lemon-flavored, we have $\frac{96}{100}$. The second part is the probability of selecting another non-Lemon flavored Jolly Rancher knowing that the first one picked was not Lemon-flavored and that it was not put back in the bag. Since our sample space has been reduced, we have $\frac{95}{99}$. Finally, the third part of the product is the probability of selecting a Lemon-flavored Jolly Rancher given what we know about the two previous selections. Our sample space is now down to 98 Jolly Ranchers, with all 4 Lemon-flavored Jolly Ranchers still available. Therefore, our third probability is $\frac{4}{98}$. Now that we have each part of the product, we know:

\[
P(\text{non-Lemon and non-Lemon and Lemon}) = \frac{96}{100} \times \frac{95}{99} \times \frac{4}{98} = \frac{36,480}{970,200}
\]

At this stage, we need to consider the other ways in which exactly one Lemon-flavored Jolly Rancher can be selected. We assumed earlier that the Lemon-flavored Jolly Rancher would be picked last. Fortunately, there are only two other cases to consider: the Lemon-flavored Jolly Rancher is picked first or it is picked second (we detailed these two cases in the previous problem). If we approach these two cases like we did the first, the probabilities are:

\[
P(\text{Lemon and non-Lemon and non-Lemon}) = \frac{4}{100} \times \frac{96}{99} \times \frac{95}{98}
\]
Notice anything about these products? They are equivalent! In fact, we could have saved some time and effort. There are \[
\frac{3!}{2!1!} = 3
\] ways to permute the three events in this experiment. Since the probability is the same in each case, we have:

\[
3 \left( \frac{96}{100} \times \frac{95}{99} \times \frac{4}{98} \right) = 3 \left( \frac{36,480}{970,200} \right) = 0.112801484 \approx 0.113
\]

4. We want the probability that a player selects no Lemon-flavored Jolly Ranchers. If we go back to using combinations, we can model this situation like we did in the last question. We already know the size of the sample space (161,700). The challenge is to determine how many of the 161,700 possible selections have no Lemon-flavored Jolly Ranchers in them. This is equivalent to finding the number of ways a player can select three non-Lemon flavored Jolly Ranchers from a set of 96 non-Lemon flavored Jolly Ranchers:

\[
_{96}C_3 = \frac{96!}{3!93!} = 142,880
\]

Therefore, our probability is:

\[
\frac{_{96}C_3}{_{100}C_3} = \frac{142,880}{161,700} = 0.883611626 \approx 0.884
\]

A player has about an 88.4% chance of selecting no Lemon-flavored Jolly Ranchers on their turn.

5. We want the probability that a player selects at least one Lemon-flavored Jolly Rancher. We can take advantage of what we know about complementary events!

\[
P(\text{at least one}) = 1 - P(\text{none})
\]

If we apply this idea, we have:

\[
P(\text{at least one Lemon}) = 1 - P(\text{no Lemons}) = 1 - \frac{142,880}{161,700} = 0.116388374 \approx 0.116
\]

A player has about an 11.6% chance of selecting at least one Lemon-flavored Jolly Rancher. In other words, a player has about an 11.6% chance of winning the new glove.*

*It is interesting to note that a player’s chance of selecting at least one Lemon-flavored Jolly Rancher is only three tenths of a percent higher than their chance of selecting exactly one Lemon-flavored Jolly Rancher.