Linear and Absolute Value Inequalities

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Definition: Union and Intersection

Let $A$ and $B$ be two sets.

The union of $A$ and $B$, denoted $A \cup B$ is the set of all elements that are members of $A$, or $B$, or both.

The intersection of $A$ and $B$, denoted $A \cap B$ is the set of all elements that are members of both $A$ and $B$. 
Example 1.

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Determine both $A \cup B$ and $A \cap B$. 

Solution.

$A \cup B = \{1, 2, 3, 4, 6\}$

$A \cap B = \{2\}$
Example 1.

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Determine both $A \cup B$ and $A \cap B$.

**Solution.**

- $A \cup B = \{1, 2, 3, 4, 6\}$
Example 1.

Let \( A = \{1, 2, 3\} \) and \( B = \{2, 4, 6\} \). Determine both \( A \cup B \) and \( A \cap B \).

**Solution.**

- \( A \cup B = \{1, 2, 3, 4, 6\} \)
- \( A \cap B = \{2\} \)
Example 2.

Let $B = \{2, 4, 6\}$ and $C = \{1, 3, 5\}$. Determine both $B \cup C$ and $B \cap C$.

Solution.

$B \cup C = \{1, 2, 3, 4, 5, 6\}$

$B \cap C = \emptyset$
Example 2.

Let $B = \{2, 4, 6\}$ and $C = \{1, 3, 5\}$. Determine both $B \cup C$ and $B \cap C$.

**Solution.**

- $B \cup C = \{1, 2, 3, 4, 5, 6\}$
Example 2.

Let $B = \{2, 4, 6\}$ and $C = \{1, 3, 5\}$. Determine both $B \cup C$ and $B \cap C$.

**Solution.**

- $B \cup C = \{1, 2, 3, 4, 5, 6\}$
- $B \cap C = \emptyset$
Example 3.

Let $D = \{x \mid 0 < x < 4\}$ and $E = \{x \mid 2 < x < 6\}$. Determine both $D \cup E$ and $D \cap E$. 

Solution.

$D \cup E = \{x \mid 0 < x < 6\}$

$D \cap E = \{x \mid 2 < x < 4\}$
Example 3.

Let $D = \{x \mid 0 < x < 4\}$ and $E = \{x \mid 2 < x < 6\}$. Determine both $D \cup E$ and $D \cap E$.

**Solution.**

- $D \cup E = \{x \mid 0 < x < 6\}$
Example 3.

Let \( D = \{x \mid 0 < x < 4\} \) and \( E = \{x \mid 2 < x < 6\} \). Determine both \( D \cup E \) and \( D \cap E \).

Solution.

- \( D \cup E = \{x \mid 0 < x < 6\} \)
- \( D \cap E = \{x \mid 2 < x < 4\} \)
For any real numbers $a$ and $b$, the following are sets written in *interval notation*.

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$
Write the following sets in interval notation.

- \( \{x \mid -3 \leq x < 5\} \)
- \( \{x \mid 7 < x \leq 10\} \)
Example 4.

Write the following sets in interval notation.

- \( \{ x \mid -3 \leq x < 5 \} \)
- \( \{ x \mid 7 < x \leq 10 \} \)

Solution.

- \([-3, 5)\)
- \((7, 10]\)
Example 5.

Write the following intervals in set notation.

- (3, 8)
- [−2, 5]
Example 5.

Write the following intervals in set notation.

- $(3, 8)$
- $[-2, 5]$

Solution.

- $\{x \mid 3 \leq x < 8\}$
- $\{x \mid -2 \leq x \leq 5\}$
Unbounded Intervals

\[(a, \infty) = \{x \mid x > a\}\]

\[[a, \infty) = \{x \mid x \geq a\}\]

\[(-\infty, b) = \{x \mid x < b\}\]

\[(-\infty, b] = \{x \mid x \leq b\}\]
Example 6.

Write the following sets in interval notation.

- \( \{x \mid x \geq -2\} \)
- \( \{x \mid x < -2\} \)
Example 6.

Write the following sets in interval notation.

- \( \{ x \mid x \geq -2 \} \)
- \( \{ x \mid x < -2 \} \)

**Solution.**

- \([ -2, \infty ) \)
- \((-\infty, -2) \)
Example 7.

Let $A = (1, 4)$ and $B = (2, 5)$. Determine both $A \cup B$ and $A \cap B$. 

Solution.

$A \cup B = (1, 5)$

$A \cap B = (2, 4)$
Example 7.

Let $A = (1, 4)$ and $B = (2, 5)$. Determine both $A \cup B$ and $A \cap B$.

Solution.

- $A \cup B = (1, 5)$
Example 7.

Let \( A = (1, 4) \) and \( B = (2, 5) \). Determine both \( A \cup B \) and \( A \cap B \).

Solution.

- \( A \cup B = (1, 5) \)
- \( A \cap B = (2, 4) \)
Example 8.

Let $B = (2, 5)$ and $C = [3, 6]$. Determine both $B \cup C$ and $B \cap C$. 
Example 8.

Let $B = (2, 5)$ and $C = [3, 6]$. Determine both $B \cup C$ and $B \cap C$.

**Solution.**

- $B \cup C = (2, 6]$
Example 8.

Let $B = (2, 5)$ and $C = [3, 6]$. Determine both $B \cup C$ and $B \cap C$.

Solution.

- $B \cup C = (2, 6]$
- $B \cap C = [3, 5]$
Example 9.

Let \( D = (0, 4] \) and \( E = [5, 9) \). Determine both \( D \cup E \) and \( D \cap E \).
Example 9.

Let \( D = (0, 4] \) and \( E = [5, 9) \). Determine both \( D \cup E \) and \( D \cap E \).

Solution.

\[ D \cup E = (0, 4] \cup [5, 9) \]
Example 9.

Let $D = (0, 4]$ and $E = [5, 9)$. Determine both $D \cup E$ and $D \cap E$.

Solution.

- $D \cup E = (0, 4] \cup [5, 9)$
- $D \cap E = \emptyset$
Example 10.

Solve the linear inequality $3x - 7 < 5$. Write the answer in interval notation.

Solution.

$3x - 7 < 5$

$3x < 12$

$x < 4$

The solution is $(-\infty, 4)$. 
Example 10.

Solve the linear inequality $3x - 7 < 5$. Write the answer in interval notation.

**Solution.**

\[
3x - 7 < 5
\]

\[
3x < 12
\]

\[
x < 4
\]

The solution is $(-\infty, 4)$. 
Solve the linear inequality $-2x - 7 \leq 19$. Write the answer in interval notation.
Example 11.

Solve the linear inequality \(-2x - 7 \leq 19\). Write the answer in interval notation.

Solution.

\[-2x - 7 \leq 19\]

\[-2x \leq 26\]

\[x \geq -13\]

The solution is \([-13, \infty)\).
Example 12.

Solve the linear inequality $1 < 4x - 3 \leq 11$. Write the answer in interval notation.
Solve the linear inequality $1 < 4x - 3 \leq 11$. Write the answer in interval notation.

**Solution.**

$$1 < 4x - 3 \leq 11$$

$$4 < 4x \leq 14$$

$$1 < x \leq \frac{7}{2}$$

The solution is $\left(1, \frac{7}{2}\right]$. 
Example 13.

Solve the linear inequality $-2 \leq \frac{1 - 2x}{3} \leq 3$. Write the answer in interval notation.

Solution.

$-2 \leq \frac{1 - 2x}{3} \leq 3$

$-6 \leq 1 - 2x \leq 9$

$-7 \leq -2x \leq 8$

$\frac{7}{2} \geq x \geq -4$

The solution is $[-4, \frac{7}{2}]$. 
Example 13.

Solve the linear inequality $-2 \leq \frac{1 - 2x}{3} \leq 3$. Write the answer in interval notation.

**Solution.**

\[
-2 \leq \frac{1 - 2x}{3} \leq 3
\]

\[
-6 \leq 1 - 2x \leq 9
\]

\[
-7 \leq -2x \leq 8
\]

\[
\frac{7}{2} \geq x \geq -4
\]

The solution is $\left[-4, \frac{7}{2}\right]$. 

Joseph Lee  
Linear and Absolute Value Inequalities
Example 14.

Solve $|x| < 4$.
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Solve $|x| < 4$.

**Solution.** Notice $x = 5$ is not a solution to this inequality. Clearly, $x < 4$ must hold for this inequality to be true. However, also notice $x = -5$ is not a solution as well. The inequality $x > -4$ must also hold. Combining these two inequalities, we may simply state

$$-4 < x < 4.$$ 

Thus, our solution is the interval $(-4, 4)$. 
For any nonnegative value \( k \), the inequality \( |x| < k \) may be expressed as

\[-k < x < k.\]
For any nonnegative value $k$, the inequality $|x| < k$ may be expressed as

$$-k < x < k.$$
Example 15.

Solve $|x| > 4$.

Solution. Notice this inequality holds for all $x > 4$. However, the inequality also holds for all $x < -4$. The solution is $(-\infty, -4) \cup (4, \infty)$.
Solve $|x| > 4$.

**Solution.** Notice this inequality holds for all $x > 4$. However, the inequality also holds for all $x < -4$. The solution is $(-\infty, -4) \cup (4, \infty)$. 
For any nonnegative value \( k \), the inequality \( |x| > k \) may be satisfied by either
\[
x > k \quad \text{or} \quad x < -k.
\]
For any nonnegative value $k$, the inequality $|x| > k$ may be satisfied by either
\[ x > k \quad \text{or} \quad x < -k. \]
Similarly, for $|x| \geq k$, we know $x \geq k$ or $x \leq -k$. 
Example 16.

Solve $|x + 8| \leq 2$. 

Solution. From observation 1, we should consider the double inequality 

$$-2 \leq x + 8 \leq 2.$$ 

Then 

$$-10 \leq x \leq -6.$$ 

The solution is $[-10, -6]$. 

Joseph Lee

Linear and Absolute Value Inequalities
Example 16.

Solve $|x + 8| \leq 2$.

**Solution.** From observation 1, we should consider the double inequality $-2 \leq x + 8 \leq 2$.

\[-2 \leq x + 8 \leq 2\]

\[-10 \leq x \leq -6\]

The solution is $[-10, -6]$. 
Example 17.

Solve $|6x + 2| \geq 2$. 

Solution.

From observation 2, we should consider two inequalities:

$6x + 2 > 2$ and $6x + 2 < -2$.

$6x > 0$ and $6x < -4$.

$x > 0$ and $x < -\frac{4}{3}$.

The solution is $(-\infty, -\frac{4}{3}] \cup [0, \infty)$. 

Joseph Lee

Linear and Absolute Value Inequalities
Example 17.

Solve $|6x + 2| \geq 2$.

**Solution.** From observation 2, we should consider two inequalities: $6x + 2 > 2$ and $6x + 2 < -2$.

\[
\begin{align*}
6x + 2 & > 2 \\
6x & > 0 \\
x & > 0
\end{align*} \quad \begin{align*}
6x + 2 & < -2 \\
6x & < -4 \\
x & < -\frac{2}{3}
\end{align*}
\]

The solution is \( (-\infty, -\frac{2}{3}] \cup [0, \infty) \).
Example 18.

Solve $|4 - x| < 8$. 

Solution.

From observation 1, we should consider the double inequality 

$-8 < 4 - x < 8$. 

$-8 < 4 - x < 8$ 

$-12 < -x < 4$ 

$x > -12$ and $x > -4$ 

The solution is $(-4, 12)$. 

Joseph Lee  
Linear and Absolute Value Inequalities
Example 18.

Solve $|4 - x| < 8$.

**Solution.** From observation 1, we should consider the double inequality $-8 < 4 - x < 8$.

$$-8 < 4 - x < 8$$

$$-12 < -x < 4$$

$$12 > x > -4$$

The solution is $(-4, 12)$. 
Example 19.

Solve $|1 - 7x| > 13$. 

Solution. From observation 2, we should consider two inequalities:

$1 - 7x > 13$ and $1 - 7x < -13$.

For $1 - 7x > 13$,

$1 - 7x > 13$

Subtract 1 from both sides:

$-7x > 12$

Divide both sides by -7 (remember to reverse the inequality):

$x < -\frac{12}{7}$

For $1 - 7x < -13$,

$1 - 7x < -13$

Subtract 1 from both sides:

$-7x < -14$

Divide both sides by -7 (remember to reverse the inequality):

$x > 2$

The solution is $(-\infty, -\frac{12}{7}) \cup (2, \infty)$. 

Joseph Lee
Linear and Absolute Value Inequalities
Example 19.

Solve $|1 - 7x| > 13$.

**Solution.** From observation 2, we should consider two inequalities: $1 - 7x > 13$ and $1 - 7x < -13$.

\[
\begin{align*}
1 - 7x &> 13 & 1 - 7x &< -13 \\
-7x &> 12 & -7x &< -14 \\
x &< -\frac{12}{7} & x &> 2
\end{align*}
\]

The solution is $\left(-\infty, -\frac{12}{7}\right) \cup (2, \infty)$. 
Example 20.

Solve $|x - 3| > -2$. 

Solution. Notice this inequality holds for all values of $x$ as $|x - 3| > 0$. Thus, the solution is $(-\infty, \infty)$. 

Joseph Lee
Linear and Absolute Value Inequalities
Example 20.

Solve $|x - 3| > -2$.

**Solution.** Notice this inequality holds for all values of $x$ as $|x - 3| > 0$. Thus, the solution is $(-\infty, \infty)$. 

Joseph Lee

Linear and Absolute Value Inequalities
For any negative value $k$, the inequality $|x| > k$ holds for any value of $x$. 
Example 21.

Solve $|3x + 2| < -5$. 

Solution. Notice this inequality fails for all values of $x$ as $|3x + 2| > 0$. Thus, there is no solution.
Example 21.

Solve $|3x + 2| < -5$.

**Solution.** Notice this inequality fails for all values of $x$ as $|3x + 2| > 0$. Thus, there is no solution.
Observation 6

For any negative value \( k \), the inequality \( |x| < k \) has no solution.