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Preface

I have taught this course at least four or five times in the past. Most recently, I taught this course during the Fall 2012. During the Winter 2012-2013, I started to compile my lecture notes for a different course – Intermediate Algebra – into a nice format for my students. My goal is to create a set of lecture notes for College Algebra that are similar to the lecture notes I have created for Intermediate Algebra. As with my previous effort, I expect that this effort will span over multiple quarters and will remain quite unfinished through Summer 2013. However, I teach this summer session with hopes that I will be able to get a start on making these notes for this course and future College Algebra sections.

The point I am trying to convey is that this set of lecture notes will remain a work in-progress throughout the summer and beyond. Where I do not have lecture notes typed out in advance, I will lecture not very differently than I did the first four or five times that I taught the course. My goal will remain though that eventually all of my lectures are typed out nicely for my students. Unfortunately, this goal will not be realized for the summer. Please use the notes I have compiled to help you as you work through the material of the course. Even in places where these notes are somewhat complete, I would encourage the student to also read the corresponding section in the textbook.

Many people view College Algebra to be one of the most difficult math courses offered at any college. To be successful in the course, the student not only must take good notes during lecture, but the student should also read his or her textbook. Most importantly, however, the student should complete all of the required homework exercises. Please understand that completing the homework for each section will take a considerable amount of time.

Tips for Success

1. Be in class every day – on time – and be prepared to work hard to understand difficult algebraic ideas. Simply attending class passively will not be sufficient. The student should take notes and ask for clarifications when necessary.

2. Complete your entire homework assignment each night, and check your answers in the back of the book as you go. Homework assistance can be received during my scheduled office hours or at the math center. Please take advantages of these opportunities if necessary.

3. Have a positive attitude toward the class. I enjoy mathematics more than anyone, so clearly you might not share my enthusiasm. Please understand though that I will enjoy lecturing each day, whether you enjoy the mathematics or not. Keeping a positive attitude, especially as we explore difficult concepts, will only help you be successful.
1 Preliminaries

1.1 Rational Expressions

Definition 1. Rational Expression. A rational expression is a quantity described by numbers or variables and the operations of addition, subtraction, multiplication, and division.

Examples of Rational Expressions

\[ x^2 + 3x - 4, \quad \frac{x}{x + 3}, \quad \frac{x^2 - 8x + 5}{3x + 5} \]

When a person refers to a rational expression, often times he or she will be referring to something of the form the second two – the fractions.

Definition 2. Domain. The domain of a rational expression is the set of values of the variable for which the rational expression is defined.

Example 1. Consider the rational expression

\[ \frac{x}{x + 3}. \]

(a) The value \( x = 1 \) is in the domain, since \( \frac{x}{x + 3} = \frac{1}{1 + 3} = \frac{1}{4}. \)

(b) The value \( x = -1 \) is in the domain, since \( \frac{x}{x + 3} = \frac{-1}{-1 + 3} = -\frac{1}{2}. \)

(c) The value \( x = 0 \) is in the domain, since \( \frac{x}{x + 3} = \frac{0}{0 + 3} = 0. \)

(d) The value \( x = -3 \) is not in the domain, since \( \frac{x}{x + 3} = \frac{-3}{-3 + 3} = -\frac{3}{0}, \) which is undefined.

A rational expression will be defined as long as the denominator does not equal zero.

Example 2. State the domain of each rational expression.

(a) \( \frac{x}{x + 3} \)

We are only concerned that the denominator \( x + 3 \) does not equal 0.

\[ x + 3 \neq 0 \]

\[ x \neq -3 \]

The domain is all real numbers except \( x = -3. \)

(b) \( \frac{2x + 3}{3x - 2} \)
We are only concerned that the denominator $3x - 2$ does not equal 0.

\[
3x - 2 \neq 0 \\
3x \neq 2 \\
x \neq \frac{2}{3}
\]

The domain is all real numbers except $x = \frac{2}{3}$.

(c) \[
\frac{x - 4}{x^2 + 5x + 6}
\]

We are only concerned that the denominator $x^2 + 5x + 6$ does not equal 0.

\[
x^2 + 5x + 6 \neq 0 \\
(x + 2)(x + 3) \neq 0 \\
x \neq -2, \ x \neq -3
\]

The domain is all real numbers except $x = -2$ and $x = -3$.

(d) \[
\frac{x^2 + 8x + 7}{x^2 + 1}
\]

We are only concerned that the denominator $x^2 + 1$ does not equal 0. In fact, for any value of $x$, $x^2 + 1 \neq 0$. Thus, the domain is all real numbers.

It is important to note the domain of each rational expression we encounter. In previous math courses, you may have done some really bad things. Observe what I have below is really bad:

\[
\frac{3x + 6}{x + 2} = \frac{3(x + 2)}{x + 2} = 3 \quad \leftarrow \text{BAD}
\]

Is $\frac{3x + 6}{x + 2} = 3$? Let’s check:

(a) If $x = 1$, then $\frac{3x + 6}{x + 2} = \frac{3(1) + 6}{1 + 2} = \frac{9}{3} = 3$.

(b) If $x = 0$, then $\frac{3x + 6}{x + 2} = \frac{3(0) + 6}{0 + 2} = \frac{6}{2} = 3$.

(c) If $x = -2$, then $\frac{3x + 6}{x + 2} = \frac{3(-2) + 6}{-2 + 2} = \frac{0}{0}$ which is undefined (not 3).

No, $\frac{3x + 6}{x + 2}$ is not equal to 3. It is only equal to 3 if $x$ is not $-2$. We call this a domain restriction.

Here is the correct way to express the above rational expression:

\[
\frac{3x + 6}{x + 2} = \frac{3(x + 2)}{x + 2} = 3, \quad x \neq -2 \quad \leftarrow \text{GOOD}
\]

(a) \( \frac{4x - 8}{x - 2} \)

\[
\frac{4x - 8}{x - 2} = \frac{4(x - 2)}{x - 2} = \frac{x - 2}{x - 2} \cdot 4
\]

\[= 4, \quad x \neq 2\]

(b) \( \frac{x - 3}{x^2 - 5x + 6} \)

\[
\frac{x - 3}{x^2 - 5x + 6} = \frac{x - 3}{(x - 2)(x - 3)}
\]

\[= \frac{x - 3}{x - 3} \cdot \frac{1}{x - 2}
\]

\[= \frac{1}{x - 2}, \quad x \neq 2, 3\]

(c) \( \frac{x^2 - 14x + 49}{x^2 - 6x - 7} \)

\[
\frac{x^2 - 14x + 49}{x^2 - 6x - 7} = \frac{(x - 7)^2}{(x - 7)(x + 1)}
\]

\[= \frac{x - 7}{x - 7} \cdot \frac{x - 7}{x + 1}
\]

\[= \frac{x - 7}{x + 1}, \quad x \neq -1, 7\]

**Example 4.** Multiply. State any domain restrictions.

(a) \( \frac{2x}{3x + 1} \cdot \frac{3x^2 - 5x - 2}{4x^2 + 8x} \)

\[
\frac{2x}{3x + 1} \cdot \frac{3x^2 - 5x - 2}{4x^2 + 8x} = \frac{2x}{3x + 1} \cdot \frac{(3x + 1)(x - 2)}{4x(x + 2)}
\]

\[= \frac{2x \cdot 3x + 1}{2x \cdot 3x + 1} \cdot \frac{x - 2}{2(x + 2)}
\]

\[= \frac{x - 2}{2(x + 2)}, \quad x \neq -2, -\frac{1}{3}, 0\]
(b) \[ \frac{x^2 + x - 6}{4 - x^2} \cdot \frac{x^2 + 4x + 4}{x^2 + 4x + 3} \]

\[ \frac{x^2 + x - 6}{4 - x^2} \cdot \frac{x^2 + 4x + 4}{x^2 + 4x + 3} = \frac{(x + 3)(x - 2)}{(2 - x)(2 + x)} \cdot \frac{(x + 2)^2}{(x + 3)(x + 1)} \]

\[ = \frac{x + 3}{x + 3} \cdot \frac{x - 2}{2 - x} \cdot \frac{x + 2}{2 + x} \cdot \frac{x + 2}{x + 1} \]

\[ = 1 \cdot (-1) \cdot \frac{x + 2}{x + 1}, \quad x \neq -3, -2, -1, 2 \]

\[ = -\frac{x + 2}{x + 1}, \quad x \neq -3, -2, -1, 2 \]

**Example 5.** Divide. State any domain restrictions.

\[ \frac{2x^2 - 9x - 5}{4x^2 - 1} \div \frac{2x^2 - 13x + 15}{4x^2 - 8x + 3} \]

\[ \frac{2x^2 - 9x - 5}{2x^2 - 13x + 15} \div \frac{4x^2 - 1}{4x^2 - 8x + 3} = \frac{2x^2 - 9x - 5}{2x^2 - 13x + 15} \cdot \frac{4x^2 - 8x + 3}{4x^2 - 1} \]

\[ = \frac{(2x + 1)(x - 5)}{(2x - 3)(x - 5)} \cdot \frac{(2x - 3)(2x - 1)}{(2x - 1)(2x + 1)} \]

\[ = \frac{2x + 1}{2x + 1} \cdot \frac{x - 5}{x - 5} \cdot \frac{2x - 3}{2x - 3} \cdot \frac{2x - 1}{2x - 1} \]

\[ = 1, \quad x \neq -\frac{1}{2}, \frac{3}{2}, 5, \]

**Example 6.** Add. State any domain restrictions.

(a) \[ \frac{x^2 - 5x}{x^2 - 7x + 12} + \frac{3x - 3}{x^2 - 7x + 12} \]

\[ \frac{x^2 - 5x}{x^2 - 7x + 12} + \frac{3x - 3}{x^2 - 7x + 12} = \frac{x^2 - 5x + 3x - 3}{x^2 - 7x + 12} \]

\[ = \frac{x^2 - 2x - 3}{x^2 - 7x + 12} \]

\[ = \frac{(x - 3)(x + 1)}{(x - 3)(x - 4)} \]

\[ = \frac{x + 1}{x - 4}, \quad x \neq 3, 4 \]
(b) \[ \frac{x - 1}{x^2 + x - 20} + \frac{x + 3}{x^2 - 5x + 4} \]

\[ \frac{x - 1}{x^2 + x - 20} + \frac{x + 3}{x^2 - 5x + 4} = \frac{x - 1}{(x + 5)(x - 4)} + \frac{x + 3}{(x - 1)(x - 4)} \]

\[ = \frac{x - 1}{(x + 5)(x - 4)} \cdot \frac{x - 1}{x - 1} + \frac{x + 3}{(x - 1)(x - 4)} \cdot \frac{x + 5}{x + 5} \]

\[ = \frac{x^2 - 2x + 1}{(x + 5)(x - 4)(x - 1)} + \frac{x^2 + 8x + 15}{(x + 5)(x - 4)(x - 1)} \]

\[ = \frac{2x^2 + 6x + 16}{(x + 5)(x - 4)(x - 1)}, \quad x \neq -5, 1, 4 \]

**Example 7.** Subtract. State any domain restrictions.

\[ \frac{4}{x + 2} - \frac{x - 26}{x^2 - 3x - 10} \]

\[ \frac{4}{x + 2} - \frac{x - 26}{x^2 - 3x - 10} = \frac{4}{x + 2} - \frac{x - 26}{(x - 5)(x + 2)} \]

\[ = \frac{4}{x + 2} \cdot \frac{x - 5}{x - 5} - \frac{x - 26}{(x - 5)(x + 2)} \]

\[ = \frac{4x - 20}{(x - 5)(x + 2)} - \frac{x - 26}{(x - 5)(x + 2)} \]

\[ = \frac{(4x - 20) - (x - 26)}{(x - 5)(x + 2)} \]

\[ = \frac{4x - 20 - x + 26}{(x - 5)(x + 2)} \]

\[ = \frac{3x + 6}{(x - 5)(x + 2)} \]

\[ = \frac{3(x + 2)}{(x - 5)(x + 2)} \]

\[ = \frac{3}{x - 5}, \quad x \neq -2, 5 \]

**Example 8.** Simplify the complex rational expression. State any domain restrictions.
(a) \[ \frac{1 - \frac{2}{x}}{1 - \frac{4}{x^2}} \]

\[
1 - \frac{2}{x} \quad 1 - \frac{4}{x^2} = \left( \frac{1 - \frac{2}{x}}{1 - \frac{4}{x^2}} \right) \cdot \frac{x^2}{x^2}
\]

\[
= \frac{x^2 - 2x}{x^2 - 4}
\]

\[
= \frac{x(x - 2)}{(x - 2)(x + 2)}
\]

\[
= \frac{x}{x + 2}, \quad x \neq -2, 0, 2
\]

(b) \[ \frac{x}{x + 3} + \frac{2}{x + \frac{2}{x + 3}} \]

\[
\frac{x}{x + 3} + 2 = \left( \frac{x}{x + 3} + \frac{2}{x + \frac{2}{x + 3}} \right) \cdot \frac{x + 3}{x + 3}
\]

\[
= \frac{x + 2(x + 3)}{x(x + 3) + 2}
\]

\[
= \frac{x + 2x + 6}{x^2 + 3x + 2}
\]

\[
= \frac{3x + 6}{(x + 1)(x + 2)}, \quad x \neq -3, -2, -1
\]

(c) \[ \frac{1 + \frac{6}{x} + \frac{9}{x^2}}{1 - \frac{1}{x} - \frac{12}{x^2}} \]

\[
\frac{1 + \frac{6}{x} + \frac{9}{x^2}}{1 - \frac{1}{x} - \frac{12}{x^2}} = \left( \frac{1 + \frac{6}{x} + \frac{9}{x^2}}{1 - \frac{1}{x} - \frac{12}{x^2}} \right) \cdot \frac{x^2}{x^2}
\]

\[
= \frac{x^2 + 6x + 9}{x^2 - x - 12}
\]

\[
= \frac{(x + 3)^2}{(x - 4)(x + 3)}
\]

\[
= \frac{x + 3}{x - 4}, \quad x \neq -3, 0, 4
\]
1.2 Quadratic Equations

Definition 1. Quadratic Equation. A quadratic equation is an equation that can be written as
\[ ax^2 + bx + c = 0 \]
where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \).

Definition 2. Zero Factor Property. If \( a \cdot b = 0 \), then \( a = 0 \) or \( b = 0 \).

Example 1. Solve using the zero factor property.

(a) \( x^2 - 5x + 6 = 0 \)
\[ x^2 - 5x + 6 = 0 \]
\[ (x - 2)(x - 3) = 0 \]
Thus, by the zero factor property, either
\[ x - 2 = 0 \quad \text{or} \quad x - 3 = 0 \]
\[ x = 2 \quad \text{or} \quad x = 3 \]
The solution set is \( \{2, 3\} \).

(b) \( 3x(x - 2) = 4(x + 1) + 4 \)
\[ 3x(x - 2) = 4(x + 1) + 4 \]
\[ 3x^2 - 6x = 4x + 4 + 4 \]
\[ 3x^2 - 6x = 4x + 8 \]
\[ 3x^2 - 10x - 8 = 0 \]
\[ (3x + 2)(x - 4) = 0 \]
\[ 3x + 2 = 0 \quad x - 4 = 0 \]
\[ x = -\frac{2}{3} \quad x = 4 \]
The solution set is \( \left\{-\frac{2}{3}, 4\right\} \).

Definition 3. Square Root Property. If \( x^2 = a \), then \( x = \pm \sqrt{a} \).

Example 2. Solve using the square root property.
(a) \[ 3x^2 + 4 = 58 \]
\[ 3x^2 = 54 \]
\[ x^2 = 18 \]
\[ \sqrt{x^2} = \pm \sqrt{18} \]
\[ x = \pm 3\sqrt{2} \]

The solution set is \([-3\sqrt{2}, 3\sqrt{2}]\).

(b) \[ (x - 3)^2 = 4 \]
\[ (x - 3)^2 = 4 \]
\[ \sqrt{(x - 3)^2} = \pm \sqrt{4} \]
\[ x - 3 = \pm 2 \]
\[ x = 3 \pm 2 \]

The solution set is \(\{1, 5\}\).

(c) \[ (2x - 1)^2 = -5 \]
\[ (2x - 1)^2 = -5 \]
\[ \sqrt{(2x - 1)^2} = \pm \sqrt{-5} \]
\[ 2x - 1 = \pm i\sqrt{5} \]
\[ 2x = 1 \pm i\sqrt{5} \]
\[ x = \frac{1 \pm i\sqrt{5}}{2} \]

The solution set is \(\left\{ \frac{1 + i\sqrt{5}}{2}, \frac{1 - i\sqrt{5}}{2} \right\}\).

**Definition 4. Quadratic Formula.** For any quadratic equation \(ax^2 + bx + c = 0\),
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Example 3.** Solve using the quadratic formula.
(a) \(3x^2 - 5x - 2 = 0\)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{5 \pm \sqrt{(-5)^2 - 4(3)(-2)}}{2(3)}
\]

\[
= \frac{5 \pm \sqrt{25 + 24}}{6}
\]

\[
= \frac{5 \pm \sqrt{49}}{6} = \frac{5 \pm 7}{6}
\]

The solution set is \(\{-\frac{1}{3}, 2\}\).

(b) \(x^2 - 3x - 7 = 0\)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)}
\]

\[
= \frac{3 \pm \sqrt{9 + 28}}{2}
\]

\[
= \frac{3 \pm \sqrt{37}}{2}
\]

The solution set is \(\left\{\frac{3 + \sqrt{37}}{2}, \frac{3 - \sqrt{37}}{2}\right\}\).
1.3 Reducible Equations

In this section, we expand on the previous section to solve higher order equations and equations that are reducible to quadratic equations.

Example 1. Solve.

(a) \(x^3 - 16x = 0\)

\[\begin{align*}
\quad & x^3 - 16x = 0 \\
\quad & x(x^2 - 16) = 0 \\
\quad & x(x + 4)(x - 4) = 0 \\
\quad & x = 0 \\
\quad & x + 4 = 0 \\
\quad & x - 4 = 0 \\
\quad & x = -4 \\
\quad & x = 4
\end{align*}\]

The solution set is \{-4, 0, 4\}.

(b) \(8x^3 + 6x = 12x^2 + 9\)

\[\begin{align*}
\quad & 8x^3 + 6x = 12x^2 + 9 \\
\quad & 8x^3 - 12x^2 + 6x - 9 = 0 \\
\quad & 4x^2(2x - 3) + 3(2x - 3) = 0 \\
\quad & (2x - 3)(4x^2 + 3) = 0 \\
\quad & 2x - 3 = 0 \\
\quad & 4x^2 + 3 = 0 \\
\quad & x = \frac{3}{2} \\
\quad & x^2 = -\frac{3}{4} \\
\quad & x = \pm \sqrt{-\frac{3}{4}} \\
\quad & x = \pm \frac{i\sqrt{3}}{2}
\end{align*}\]

The solution set is \(\left\{\frac{3}{2}, \frac{i\sqrt{3}}{2}, -\frac{i\sqrt{3}}{2}\right\}\).

Example 2. Solve.

(a) \(x + 1 = \sqrt{x + 13}\).

\[\begin{align*}
\quad & (x + 1)^2 = \left(\sqrt{x + 13}\right)^2 \\
\quad & x^2 + 2x + 1 = x + 13 \\
\quad & x^2 + x - 12 = 0 \\
\quad & (x + 4)(x - 3) = 0
\end{align*}\]

We have solutions of \(x = -4\) and \(x = 3\). Checking those solutions however, we find that \(x = -4\) is extraneous, as

\[-4 + 1 \neq \sqrt{-4 + 13}.
\]

The solution set is \{3\}.
(b) \( \sqrt{x^2 - x + 3} - 1 = 2x. \)

\[
\sqrt{x^2 - x + 3} = 2x + 1
\]
\[
(\sqrt{x^2 - x + 3})^2 = (2x + 1)^2
\]
\[
x^2 - x + 3 = 4x^2 + 4x + 1
\]
\[
0 = 3x^2 + 5x - 2
\]
\[
0 = (3x - 1)(x + 2)
\]

We have solutions of \( x = \frac{1}{3} \) and \( x = -2 \). Checking those solutions however, we find that \( x = -2 \) is extraneous, as

\[
\sqrt{(-2)^2 - (-2) + 3} - 1 \neq 2(-2).
\]

The solution set is \( \{ \frac{1}{3} \} \).

(c) \( \sqrt{x} - 1 = \sqrt{2x + 2}. \)

\[
(\sqrt{x} - 1)^2 = (\sqrt{2x + 2})^2
\]
\[
x - 2\sqrt{x} + 1 = 2x + 2
\]
\[
(-2\sqrt{x})^2 = (x + 1)^2
\]
\[
4x = x^2 + 2x + 1
\]
\[
0 = x^2 - 2x + 1
\]
\[
0 = (x - 1)^2
\]

The solution, \( x = 1 \), is extraneous, as \( \sqrt{1} - 1 = 0 \neq \sqrt{2(1) + 2} = 2 \). Thus, there is no solution.

**Example 3.** Solve \( (x - 1)^{2/3} = 4. \)

\[
(x - 1)^{2/3} = 4
\]
\[
3\sqrt{(x - 1)^2} = 4
\]
\[
(x - 1)^2 = 64
\]
\[
x - 1 = \pm 8
\]
\[
x = 1 \pm 8
\]
\[
x = 9 \quad x = -7
\]

The solution set is \( \{-7, 9\} \).
Another important technique the student should be familiar with is substitution.

**Example 4.** Solve

(a) \(x^6 - 6x^3 + 9 = 0\).

Let \(u = x^3\).

\[
\begin{align*}
x^6 - 6x^3 + 9 &= 0 \\
u^2 - 6u + 9 &= 0 \\
(u - 3)^2 &= 0 \\
u &= 3 \\
x^3 &= 3 \\
x &= 3\sqrt[3]{3}
\end{align*}
\]

The solution set is \(\{3\sqrt[3]{3}\}\).

(b) \(x^{-2} + 2x^{-1} - 15 = 0\).

Let \(u = x^{-1}\).

\[
\begin{align*}
x^{-2} + 2x^{-1} - 15 &= 0 \\
u^2 + 2u - 15 &= 0 \\
(u + 5)(u - 3) &= 0 \\
u &= -5, \ u &= 3 \\
x^{-1} &= -5, \ x^{-1} &= 3 \\
x &= -\frac{1}{5}, \ x &= \frac{1}{3}
\end{align*}
\]

The solution set is \(\left\{\frac{1}{5}, \frac{1}{3}\right\}\).

(c) \(\frac{8}{(x-4)^2} - \frac{6}{x-4} + 1 = 0\).
Let $u = \frac{1}{x - 4}$.

$$\frac{8}{(x - 4)^2} - \frac{6}{x - 4} + 1 = 0$$

$$8u^2 - 6u + 1 = 0$$

$$(4u - 1)(2u - 1) = 0$$

$$u = \frac{1}{4}, \quad u = \frac{1}{2}$$

$$\frac{1}{x - 4} = \frac{1}{4}, \quad \frac{1}{x - 4} = \frac{1}{2}$$

$$x - 4 = 4, \quad x - 4 = 2$$

$$x = 8, \quad x = 6$$

The solution set is $\{6, 8\}$. 
1.4 Absolute Value Equations

In this section, we will explore solving equations that contain absolute values.

**Definition. Absolute Value.** The *absolute value* of a real number $x$ is the distance between 0 and $x$ on the real number line. The absolute value of $x$ is denoted by $|x|$.

**Example 1.** Solve $|x| = 7$.

Notice this equation has two solutions: $x = 7$ and $x = -7$.

**Observation 1.** For any nonnegative value $k$, if $|x| = k$, then $x = k$ or $x = -k$.

**Example 2.a.** Solve $|x - 3| = 2$.

From observation 1, we have to consider two cases: $x - 3 = 2$ and $x - 3 = -2$.

\[
\begin{align*}
x - 3 &= 2 \\
x &= 5
\end{align*}
\]

\[
\begin{align*}
x - 3 &= -2 \\
x &= 1
\end{align*}
\]

**Example 2.b.** Solve $|3x + 5| = 8$.

From observation 1, we have to consider two cases: $3x + 5 = 8$ and $3x + 5 = -8$.

\[
\begin{align*}
3x + 5 &= 8 \\
3x &= 3 \\
x &= 1
\end{align*}
\]

\[
\begin{align*}
3x + 5 &= -8 \\
3x &= -13 \\
x &= -\frac{13}{3}
\end{align*}
\]

**Example 3.** Solve $|x + 2| = -3$.

Notice this equation has no solution. No matter what value of $x$ we choose, $|x + 2| \geq 0$.

**Observation 2.** For any negative value $k$, the equation $|x| = k$ has no solution.

**Example 4.a.** Solve $|x + 9| - 3 = 1$.

\[
|\begin{align*}
x + 9| - 3 &= 1 \\
|x + 9| &= 4
\end{align*}
\]

From observation 1, we have to consider two cases: $x + 9 = 4$ and $x + 9 = -4$.

\[
\begin{align*}
x + 9 &= 4 \\
x &= -5
\end{align*}
\]

\[
\begin{align*}
x + 9 &= -4 \\
x &= -13
\end{align*}
\]
Example 4.b. Solve $-2|3x + 2| + 1 = 0$.

$$-2|3x + 2| + 1 = 0$$

$$-2|3x + 2| = -1$$

$$|3x + 2| = \frac{1}{2}$$

From observation 1, we have to consider two cases: $3x + 2 = \frac{1}{2}$ and $3x + 2 = -\frac{1}{2}$.

$$3x + 2 = \frac{1}{2} \quad 3x + 2 = -\frac{1}{2}$$

$$3x = -\frac{3}{2} \quad 3x = -\frac{5}{2}$$

$$x = -\frac{1}{2} \quad x = -\frac{5}{6}$$
1.5 Linear Inequalities

In this section, we solve linear inequalities. The algebra that we will use is very basic and should be review for the student. Our main focus, therefore, is not the simple algebra. Instead, this section is very important to us in learning *interval notation*. After completing this section, the student needs to be able to use interval notation, as we will use it throughout the rest of the course.

To properly define interval notation, we will first describe *set-builder notation*.

**Definition. Set.** A *set* is a collection. Members of the collection are called *elements*.

**Example 1.a.**
- \{black, white, pink\} is the set of Joseph’s three favorite colors. This set has 3 elements.
- \{1,2,3,4\} is the set of positive integers less than 5.
- \{-2,2\} is the set of solutions to the equation \(x^2 = 4\).
- \(\emptyset\) is the set containing no elements. For example, the set of real solutions to the equation \(x^2 = -4\).

The sets in Example 1.a are all finite. That is to say the lists that I give do not go on forever. We may also want to talk about sets with an infinite number of elements.

**Example 1.b.**
- \{1,2,3,4,5,\ldots\} could be the set of positive integers.
- \{3,5,7,\ldots\} could be the set of odd numbers greater than 1: \{3,5,7,9,11,13,\ldots\}
- \{3,5,7,\ldots\} could be the set of odd prime numbers: \{3,5,7,11,13,17,\ldots\}

Notice the confusion created in Example 1.b. If we write \{3,5,7,\ldots\}, the reader has no idea what we are referring to unless we state it explicitly. Set-builder notation is a method for describing a set explicitly.

**Example 1.c.** The following sets are written using set-builder notation.
- \(\{x \mid x \text{ is one of Joseph’s three favorite colors}\}\)
- \(\{x \mid x \text{ is an odd prime number}\}\)
- \(\{x \mid x \text{ is an integer } \& \ 0 < x < 5\}\)

There are two operations basic operations on sets that we want to understand: *union* (\(\cup\)) and *intersection* (\(\cap\)).

**Definition. Union.** Let \(A\) and \(B\) be two sets. The union of \(A\) and \(B\), denoted \(A \cup B\) is the set of all elements that are members of \(A\), or \(B\), or both.

**Definition. Intersection.** Let \(A\) and \(B\) be two sets. The intersection of \(A\) and \(B\), denoted \(A \cap B\) is the set of all elements that are members of both \(A\) and \(B\).

**Example 2.a.** Let \(A = \{1,2,3\}\) and \(B = \{2,4,6\}\). Determine both \(A \cup B\) and \(A \cap B\).
• $A \cup B = \{1, 2, 3, 4, 6\}$
• $A \cap B = \{2\}$

**Example 2.b.** Let $B = \{2, 4, 6\}$ and $C = \{1, 3, 5\}$. Determine both $B \cup C$ and $B \cap C$.

• $B \cup C = \{1, 2, 3, 4, 5, 6\}$
• $B \cap C = \emptyset$

**Example 2.c.** Let $D = \{x \mid 0 < x < 4\}$ and $E = \{x \mid 2 < x < 6\}$. Determine both $D \cup E$ and $D \cap E$.

• $D \cup E = \{x \mid 0 < x < 6\}$
• $D \cap E = \{x \mid 2 < x < 4\}$

Consider $\{x \mid 2 < x < 4\}$ from above. How many elements does this set have? We want to notice this set has an infinite number of elements. The set contains: $3, 2.5, 2.1, 2.01, 2.001, \ldots$. Clearly, we cannot write out all the numbers between 2 and 4. Thus, we must either use set-builder notation, or we must develop an easy method for writing all the numbers between 2 and 4. The method that we will use is called *interval notation*.

**Definition. Interval Notation.** For any real numbers $a$ and $b$, the following are sets written in interval notation.

\[
(a, b) = \{x \mid a < x < b\}
\]

\[
(a, b] = \{x \mid a < x \leq b\}
\]

\[
[a, b) = \{x \mid a \leq x < b\}
\]

\[
[a, b] = \{x \mid a \leq x \leq b\}
\]

So if we want to write every number between 2 and 4, we may simply write $(2, 4)$. This is the set contain $3, 3.1, 3.14, \text{etc.}$

**Example 3.** Write the following sets in interval notation.

• $\{x \mid -3 \leq x < 5\}$
• $\{x \mid 7 < x \leq 10\}$

Notice the first set is the set of all numbers between $-3$ and 5, including $-3$ but not including 5. We write this set as $[-3, 5)$. The second set is the set of all number between 7 and 10, not including 7 but including 10. We write this set as $(7, 10]$.

What if we just wanted to write every number bigger than 2? Here we use the idea of infinity, $\infty$. Infinity is more than every real number. Thus, we may write $(2, \infty)$. Notice we write $(2, \infty)$ and not $(2, \infty]$ as we do not want to include infinity.

**Example 4.** Write the following sets in interval notation.

• $\{x \mid x \geq -2\}$
• \{ x \mid x < -2 \}

The first set asks for all numbers greater than or equal to \(-2\). We may write this as \([-2, \infty)\). The second set is all numbers strictly less than \(-2\). In interval notation, we write this as \((-\infty, -2)\).

Let us re-visit our operations on sets.

**Example 5.a.** Let \(A = (1, 4)\) and \(B = (2, 5)\). Determine both \(A \cup B\) and \(A \cap B\).
- \(A \cup B = (1, 5)\)
- \(A \cap B = (2, 4)\)

**Example 5.b.** Let \(B = (2, 5)\) and \(C = [3, 6]\). Determine both \(B \cup C\) and \(B \cap C\).
- \(B \cup C = (2, 6]\)
- \(B \cap C = [3, 5]\)

**Example 5.c.** Let \(D = (0, 4]\) and \(E = [5, 9)\). Determine both \(D \cup E\) and \(D \cap E\).
- \(D \cup E = (0, 4]\ \cup \ [5, 9)\)
- \(D \cap E = \emptyset\)

Again, the intention of this section is for the student to master interval notation, as we will use it through the rest of the course. The rest of the section provides some simple algebraic exercises, and the reader is asked to use interval notation.

**Example 6.a.** Solve the linear inequality \(3x - 7 < 5\). Write the answer in interval notation.

\[
3x - 7 < 5 \\
3x < 12 \\
x < 4
\]

The solution is \((-\infty, 4)\).

**Example 6.b.** Solve the linear inequality \(-2x - 7 \leq 19\). Write the answer in interval notation.

\[
-2x - 7 \leq 19 \\
-2x \leq 26 \\
x \geq -13
\]

The solution is \([-13, \infty)\).
Example 6.c. Solve the linear inequality $1 < 4x - 3 \leq 11$. Write the answer in interval notation.

\begin{align*}
1 &< 4x - 3 \leq 11 \\
4 &< 4x \leq 14 \\
1 &< x \leq \frac{7}{2}
\end{align*}

The solution is $\left(1, \frac{7}{2}\right]$.

Example 6.d. Solve the linear inequality $-2 \leq \frac{1 - 2x}{3} \leq 3$. Write the answer in interval notation.

\begin{align*}
-2 &\leq \frac{1 - 2x}{3} \leq 3 \\
-6 &\leq 1 - 2x \leq 9 \\
-7 &\leq -2x \leq 8 \\
\frac{7}{2} &\geq x \geq -4
\end{align*}

The solution is $\left[-4, \frac{7}{2}\right]$. 
1.6 Absolute Value Inequalities

In this section, we will explore solving inequalities that contain absolute values.

Example 1. Solve $|x| < 4$.

Notice $x = 5$ is not a solution to this inequality. Clearly, $x < 4$ must hold for this inequality to be true. However, also notice $x = -5$ is not a solution as well. The inequality $x > -4$ must also hold. Combining these two inequalities, we may simply state

$$-4 < x < 4.$$ 

Thus, our solution is the interval $(-4, 4)$.

Observation 1. For any nonnegative value $k$, the inequality $|x| < k$ may be expressed as

$$-k < x < k.$$ 

Similarly, for $|x| \leq k$, we have $-k \leq x \leq k$.

Example 2. Solve $|x| > 4$.

Notice this inequality holds for all $x > 4$. However, the inequality also holds for all $x < -4$. The solution is $(-\infty, -4) \cup (4, \infty)$.

Observation 2. For any nonnegative value $k$, the inequality $|x| > k$ may be satisfied by either

$$x > k \quad \text{or} \quad x < -k.$$ 

Similarly, for $|x| \geq k$, we know $x \geq k$ or $x \leq -k$.

Example 3.a. Solve $|x + 8| \leq 2$.

From observation 1, we should consider the double inequality $-2 \leq x + 8 \leq 2$.

$$-2 \leq x + 8 \leq 2$$

$$-10 \leq x \leq -6$$

The solution is $[-10, -6]$.

Example 3.b. Solve $|6x + 2| \geq 2$.

From observation 2, we should consider two inequalities: $6x + 2 > 2$ and $6x + 2 < -2$.

$$6x + 2 > 2 \quad \quad 6x + 2 < -2$$

$$6x > 0 \quad \quad 6x < -4$$

$$x > 0 \quad \quad x < \frac{-2}{3}$$
The solution is \((-\infty, -\frac{2}{3}] \cup [0, \infty)\).

**Example 3.c.** Solve \(|4 - x| < 8\).

From observation 1, we should consider the double inequality \(-8 < 4 - x < 8\).

\[-8 < 4 - x < 8\]
\[-12 < -x < 4\]
\[12 > x > -8\]

The solution is \((-8, 12)\).

**Example 3.d.** Solve \(|1 - 7x| > 13\).

From observation 2, we should consider two inequalities: \(1 - 7x > 13\) and \(1 - 7x < -13\).

\[1 - 7x > 13\]
\[1 - 7x < -13\]
\[-7x > 12\]
\[-7x < -14\]
\[x < -\frac{12}{7}\]
\[x > 2\]

The solution is \((-\infty, -\frac{12}{7}) \cup (2, \infty)\).

**Example 4.** Solve \(|x - 3| > -2\).

Notice this inequality holds for all values of \(x\) as \(|x - 3| > 0\). Thus, the solution is \((-\infty, \infty)\).

**Observation 3.** For any negative value \(k\), the inequality \(|x| > k\) holds for any value of \(x\).

**Example 5.** Solve \(|3x + 2| < -5\).

Notice this inequality fails for all values of \(x\) as \(|3x + 2| > 0\). Thus, there is no solution.

**Observation 4.** For any negative value \(k\), the inequality \(|x| < k\) has no solution.
1.7 Non-linear Inequalities

In this section, we will explore inequalities that are nonlinear. Specifically, we will have inequalities containing higher-degree polynomials and rational expressions.

Example 1.a. Solve the non-linear equation \( x^2 - 7x + 12 = 0 \).
We know how to solve this equation:

\[
x^2 - 7x + 12 = 0
\]

\[
(x - 3)(x - 4) = 0
\]

This left side of the equation will equal zero if \( x = 3 \) or if \( x = 4 \). Thus, we have solutions of \( x = 3 \) or \( x = 4 \).

Example 1.b. Solve the non-linear inequality \( x^2 - 7x + 12 > 0 \).
Notice the similarity to the equation \( x^2 - 7x - 12 = 0 \). Let’s proceed as before:

\[
x^2 - 7x + 12 > 0
\]

\[
(x - 3)(x - 4) > 0
\]

We are looking for the left side of the equation to be positive. We know values of \( x = 3 \) and \( x = 4 \) will make the left side equal to zero. These values are called **boundary points**. To determine what values satisfy our inequality, we must look at all the regions separated by the boundary points. Thus, there are three different cases we need to consider:

- **Case (1):** If \( x \) is in the interval \((-∞, 3)\). Then:
  - \( (x - 3) \) is negative,
  - \( (x - 4) \) is negative,
  - so \( (x - 3)(x - 4) \) is positive.
  
  Thus, any \( x \) in the interval \((-∞, 3)\) is a solution.

- **Case (2):** If \( x \) is in the interval \((3, 4)\). Then:
  - \( (x - 3) \) is positive,
  - \( (x - 4) \) is negative,
  - so \( (x - 3)(x - 4) \) is negative.
  
  Thus, any \( x \) in the interval \((3, 4)\) is not a solution.

- **Case (3):** If \( x \) is in the interval \((4, ∞)\). Then:
  - \( (x - 3) \) is positive,
  - \( (x - 4) \) is positive,
  - so \( (x - 3)(x - 4) \) is positive.
  
  Thus, any \( x \) in the interval \((4, ∞)\) is a solution.
Keep in mind that \( x = 3 \) and \( x = 4 \) are not solutions to the inequality. Therefore, we may conclude our solution is \((-\infty, 3) \cup (4, \infty)\).

**Example 2.a.** Solve the non-linear inequality \( x^2 + x \leq 20 \).

We want to make one side of this inequality zero.

\[
x^2 + x \leq 20
\]

\[
x^2 + x - 20 \leq 0
\]

\[
(x + 5)(x - 4) \leq 0
\]

We are looking for the left side of the equation to be negative or equal to zero. We know values of \( x = -5 \) and \( x = 4 \) will make the left side equal to zero – these are our boundary points. Thus, there are three different cases we need to consider:

- **Case (1):** If \( x \) is in the interval \((-\infty, -5)\). Then:
  - \( (x + 5) \) is negative,
  - \( (x - 4) \) is negative,
  - so \( (x + 5)(x - 4) \) is positive.

  Thus, any \( x \) in the interval \((-\infty, -5)\) is not a solution.

- **Case (2):** If \( x \) is in the interval \((-5, 4)\). Then:
  - \( (x + 5) \) is positive,
  - \( (x - 4) \) is negative,
  - so \( (x + 5)(x - 4) \) is negative.

  Thus, any \( x \) in the interval \((-5, 4)\) is a solution.

- **Case (3):** If \( x \) is in the interval \((4, \infty)\). Then:
  - \( (x + 5) \) is positive,
  - \( (x - 4) \) is positive,
  - so \( (x + 5)(x - 4) \) is positive.

  Thus, any \( x \) in the interval \((4, \infty)\) is not a solution.

Keep in mind that \( x = -5 \) and \( x = 4 \) are solutions to the inequality. Therefore, we may conclude our solution is \([-5, 4]\).

**Example 2.b.** Solve the non-linear inequality \( 4x^2 \geq 4x + 3 \).

We want to make one side of this inequality zero.

\[
4x^2 \geq 4x + 3
\]

\[
4x^2 - 4x - 3 \geq 0
\]

\[
(2x + 1)(2x - 3) \geq 0
\]
We are looking for the left side of the equation to be positive or equal to zero. We know values of \( x = -\frac{1}{2} \) and \( x = \frac{3}{2} \) will make the left side equal to zero – these are our boundary points. Thus, there are three different cases we need to consider:

- **Case (1):** If \( x \) is in the interval \((-\infty, -\frac{1}{2})\). Then:
  - \( (2x + 1) \) is negative,
  - \( (2x - 3) \) is negative,
  - so \( (2x + 1)(2x - 3) \) is positive.

  Thus, any \( x \) in the interval \((-\infty, -\frac{1}{2})\) is a solution.

- **Case (2):** If \( x \) is in the interval \((-\frac{1}{2}, \frac{3}{2})\). Then:
  - \( (2x + 1) \) is positive,
  - \( (2x - 3) \) is negative,
  - so \( (2x + 1)(2x - 3) \) is negative.

  Thus, any \( x \) in the interval \((-\frac{1}{2}, \frac{3}{2})\) is not a solution.

- **Case (3):** If \( x \) is in the interval \((\frac{3}{2}, \infty)\). Then:
  - \( (2x + 1) \) is positive,
  - \( (2x - 3) \) is positive,
  - so \( (2x + 1)(2x - 3) \) is positive.

  Thus, any \( x \) in the interval \((\frac{3}{2}, \infty)\) is a solution.

Keep in mind that \( x = -\frac{1}{2} \) and \( x = \frac{3}{2} \) are solutions to the inequality. Therefore, we may conclude our solution is \((-\infty, -\frac{1}{2}] \cup [\frac{3}{2}, \infty)\).

**Example 3.a.** Solve the non-linear inequality \( \frac{x - 3}{x + 4} \geq 0 \).

With a rational inequality, we need to examine both the numerator and the denominator. The value \( x = 3 \) makes the numerator equal to 0, and if the numerator is 0, then the value of the expression is 0. The value \( x = -4 \) makes the denominator equal to 0, and if the denominator is 0, then the expression is undefined. The boundary points for rational inequalities include any value that makes numerator or denominator equal to 0. We now consider three different cases:

- **Case (1):** If \( x \) is in the interval \((-\infty, -4)\). Then:
  - \( (x - 3) \) is negative,
  - \( (x + 4) \) is negative,
  - so \( \frac{x - 3}{x + 4} \) is positive.

  Thus, any \( x \) in the interval \((-\infty, -4)\) is a solution.

- **Case (2):** If \( x \) is in the interval \((-4, 3)\). Then:
\( (x - 3) \) is negative,
\( (x + 4) \) is positive,
\( \frac{x - 3}{x + 4} \) is negative.

Thus, any \( x \) in the interval \((-4, 3)\) is not a solution.

- Case (3): If \( x \) is in the interval \((3, \infty)\). Then:
  \( (x - 3) \) is positive,
  \( (x + 4) \) is positive,
  \( \frac{x - 3}{x + 4} \) is positive.

Thus, any \( x \) in the interval \((3, \infty)\) is a solution.

Keep in mind that \( x = 3 \) is a solution to the inequality, but \( x = -4 \) is not a solution to the inequality (as it not in the domain of the expression). Therefore, we may conclude our solution is \((-\infty, -4) \cup [3, \infty)\).

**Example 3.b.** Solve the non-linear inequality \( \frac{2x^2 - 5x + 3}{2 - x} \geq 0 \).

\[
\frac{2x^2 - 5x + 3}{2 - x} \geq 0
\]

\[
\frac{(2x - 3)(x - 1)}{2 - x} \geq 0
\]

Our boundary points are any values that make the expression equal to zero or undefined. The values \( x = \frac{3}{2} \) and \( x = 1 \) make the expression equal to zero, and the value \( x = 2 \) make the expression undefined. We now consider four different cases:

- Case (1): If \( x \) is in the interval \((-\infty, 1)\). Then:
  \( (2x - 3) \) is negative,
  \( (x - 1) \) is negative,
  \( (2 - x) \) is positive,
  \( \frac{(2x - 3)(x - 1)}{2 - x} \) is positive.

Thus, any \( x \) in the interval \((-\infty, 1)\) is not a solution.

- Case (2): If \( x \) is in the interval \((1, \frac{3}{2})\). Then:
  \( (2x - 3) \) is negative,
  \( (x - 1) \) is positive,
  \( (2 - x) \) is positive,
  \( \frac{(2x - 3)(x - 1)}{2 - x} \) is negative.
Thus, any $x$ in the interval $(1, \frac{3}{2})$ is a solution.

- **Case (3):** If $x$ is in the interval $(\frac{3}{2}, 2)$. Then:
  - $(2x - 3)$ is positive,
  - $(x - 1)$ is positive,
  - $(2 - x)$ is positive,
  - so $\frac{(2x - 3)(x - 1)}{2 - x}$ is positive.

  Thus, any $x$ in the interval $(\frac{3}{2}, 2)$ is not a solution.

- **Case (4):** If $x$ is in the interval $(2, \infty)$. Then:
  - $(2x - 3)$ is positive,
  - $(x - 1)$ is positive,
  - $(2 - x)$ is negative,
  - so $\frac{(2x - 3)(x - 1)}{2 - x}$ is negative.

  Thus, any $x$ in the interval $(2, \infty)$ is a solution.

Keep in mind that $x = 1$ and $x = \frac{3}{2}$ are solutions to the inequality, but $x = 2$ is not a solution to the inequality (as it not in the domain of the expression). Therefore, we may conclude our solution is $[1, \frac{3}{2}] \cup (2, \infty)$.

**Example 4.** Solve the non-linear inequality $\frac{x}{x + 4} \geq 2$.

\[
\frac{x}{x + 4} \geq 2
\]

\[
\frac{x}{x + 4} - 2 \geq 0
\]

\[
\frac{x}{x + 4} - 2 \cdot \frac{x + 4}{x + 4} \geq 0
\]

\[
\frac{x}{x + 4} - \frac{2x + 8}{x + 4} \geq 0
\]

\[
\frac{x - (2x + 8)}{x + 4} \geq 0
\]

\[
\frac{x - 2x - 8}{x + 4} \geq 0
\]

\[
\frac{-x - 8}{x + 4} \geq 0
\]

Our boundary points are any values that make the expression equal to zero or undefined. The value are $x = -8$ make the expression equal to zero, and the value $x = -4$ make the expression undefined. We now consider three different cases:
• Case (1): If \( x \) is in the interval \((-\infty, -8)\). Then:
  
  \( -x - 8 \) is positive,
  \( x + 4 \) is negative,
  \( \frac{-x - 8}{x + 4} \) is negative.

  Thus, any \( x \) in the interval \((-\infty, -8)\) is not a solution.

• Case (2): If \( x \) is in the interval \((-8, -4)\). Then:
  
  \( -x - 8 \) is negative,
  \( x + 4 \) is negative,
  \( \frac{-x - 8}{x + 4} \) is positive.

  Thus, any \( x \) in the interval \((-8, -4)\) is a solution.

• Case (3): If \( x \) is in the interval \((-4, \infty)\). Then:
  
  \( -x - 8 \) is negative,
  \( x + 4 \) is positive,
  \( \frac{-x - 8}{x + 4} \) is negative.

  Thus, any \( x \) in the interval \((-4, \infty)\) is not a solution.

Since \( x = -8 \) makes the expression equal to zero, it is a solution. However, since \( x = -4 \) makes the expression undefined, it is not a solution. Thus, the solution is \([-8, -4)\).
2 Functions

2.1 Functions

Definition 1. Relation. A relation is a correspondence between two sets. Elements of the first set are called the domain. Elements of the second set are called the range. Relations are often expressed as sets of ordered pairs.

Example 1. A relation is a correspondence between two sets. Elements of the first set are called the domain. Elements of the second set are called the range.

1. \{(Joseph, turkey), (Joseph, roast beef), (Michael, ham)\} is a relation between math instructors and sandwiches they enjoy. The domain of this relation is \{Joseph, Michael\}. The range of this relation is \{turkey, roast beef, ham\}.

2. \{(1, 3), (2, 4), (−1, 1)\} is a relation of \(x\) and \(y\) values that satisfy the equation \(y = x + 2\). The domain of this relation is \{−1, 1, 2\}. The range of this relation is \{1, 3, 4\}.

3. \{(3, 5), (4, 5), (5, 5)\} is a relation of \(x\) and \(y\) values that satisfy the equation \(y = 5\).

Definition 2. Function. A function is a specific type of a relation where each element in the domain corresponds to exactly one element in the range.

Example 2. Determine which of the following relations are functions.

1. \{(Joseph, turkey), (Joseph, roast beef), (Michael, ham)\}
2. \{(1, 3), (2, 4), (−1, 1)\}
3. \{(3, 5), (4, 5), (5, 5)\}

Solution: The first relation is not a function, as Joseph is assigned to two elements in the range, turkey and roast beef. The last two relations are functions as each element in the domain is assigned to exactly one element in the range.

Example 3.a. The graph of the relation \(y = 3x - 2\) is shown below.

1. Determine the domain of the relation. \((-\infty, \infty)\)
2. Determine the range of the relation. \((-\infty, \infty)\)
3. Determine if the relation is represents a function. Yes, it is a function.
Example 3.b. The graph of the relation $x = y^2 - 1$ is shown below.

![Graph of $x = y^2 - 1$](image)

1. Determine the domain of the relation. $[-1, \infty)$
2. Determine the range of the relation. $(-\infty, \infty)$
3. Determine if the relation is represents a function. No, it is not a function.

Example 3.c. The graph of the relation $y = x^2 - 2x - 3$ is shown below.

![Graph of $y = x^2 - 2x - 3$](image)

1. Determine the domain of the relation. $(-\infty, \infty)$
2. Determine the range of the relation. $[-4, \infty)$
3. Determine if the relation is represents a function. Yes, it is a function.

Definition 3. Vertical Line Test. If any vertical line intersects the graph of a relation more than once, the relation is not a function.

Definition 4. Function Notation. If a relation of $x$ and $y$ is a function, then you may solve for $y$ and replace $y$ with $f(x)$. This notation is called function notation.

Example 4.a. Write the function $2x + 3y = 6$ in function notation.

Solution:

$$2x + 3y = 6$$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2$$

$$f(x) = -\frac{2}{3}x + 2$$
Example 4.b. Evaluate the function \( f(x) = \frac{-2}{3}x + 2 \) for the following values.

1. \( f(-3) = 4 \)
2. \( f(0) = 2 \)
3. \( f(1) = \frac{4}{3} \)
4. \( f(3) = 0 \)

Example 4.c. Evaluate the function \( f(x) = \sqrt{x} + 6 \) for the following values.

1. \( f(-2) = 2 \)
2. \( f(19) = 5 \)
3. \( f(-7) \) is undefined

Example 4.d. Evaluate the function \( f(x) = x^2 + 2x + 1 \) for the following values.

1. \( f(-x) = x^2 - 2x + 1 \)
2. \( f(x - 1) = x^2 \)
3. \( f(x + 1) = x^2 + 4x + 4 \)

Example 5.a. The graph of the function \( f \) is shown below.

![Graph of a function](image)

Evaluate.
1. \( f(-4) = -3 \)
2. \( f(-1) = 0 \)
3. \( f(1) = 2 \)
4. \( f(4) = 1 \)

**Example 5.b.** The graph of the function \( f \) is shown below.

![Graph of function f](image)

Evaluate.

1. \( f(-5) = 0 \)
2. \( f(-1) = -2 \)
3. \( f(2) = 0 \)

A function is simply a rule. Many things in life can be described by rules. If I approach a stop sign, I have to stop my car. If I eat ice cream three times a day, I will gain weight. If I quit coming to work, I will lose my job. Most people will accept these rules, but you might notice rules are sometimes more complex. If I notify my boss I am very ill and provide documentation for this illness, I will not lose my job even though I quit coming to work. If I eat ice cream for every meal, but I am training to be an Olympic swimmer, I might not gain any weight. If I approach a stop sign – and there are no cops around...

The point is: most rules we encounter are not always completely certain – they come with special cases. The same is true with functions. For example, a basic rule is if you go out to eat at a restaurant, you should tip your server 20%. However, I always leave a tip of at least $5. Here is the rule:

**Example 6.** Let \( x \) represent the cost of your meal. Calculate the tip you will leave by the following piecewise function:

\[
    f(x) = \begin{cases} 
      5 & \text{if } x \leq 25 \\ 
      0.2x & \text{if } x > 25 
    \end{cases}
\]

Notice that this function is a rule that is dependent on some condition. The functions that we will examine next are constructed in the same manner.
Example 7.a. Consider the following piecewise function:

$$f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x \geq 0 
\end{cases}$$

Evaluate the function for the following values.

1. $f(-2)$
2. $f(-1)$
3. $f(0)$
4. $f(1)$

Example 7.b. Consider the following piecewise function:

$$f(x) = \begin{cases} 
1 & \text{if } x < -1 \\
x & \text{if } -1 \leq x \leq 3 \\
2 & \text{if } x > 3 
\end{cases}$$

Evaluate the function for the following values.

1. $f(-2)$
2. $f(-1)$
3. $f(2)$
4. $f(3)$
5. $f(6)$

Example 7.c. Consider the following piecewise function:

$$f(x) = \begin{cases} 
2x + 8 & \text{if } x \leq -2 \\
x^2 & \text{if } -2 < x < 1 \\
1 & \text{if } x > 1 
\end{cases}$$

Evaluate the function for the following values.

1. $f(-2)$
2. $f(-1)$
3. $f(1)$
4. $f(4)$

Example 7.d. Consider the following piecewise function:

$$f(x) = \begin{cases} 
-x & \text{if } x < 0 \\
x & \text{if } x \geq 0 
\end{cases}$$

Evaluate the function for the following values.
1. \( f(-2) \)

2. \( f(-1) \)

3. \( f(0) \)

4. \( f(1) \)

5. \( f(2) \)

Example 7.d is really what function?

**Example 8.** Graph the functions given in Example 7.
2.2 Introductory Calculus Topics

To be clear, we are not performing any calculus in this section. However, this section resembles the spirit of calculus. We introduce increasing, decreasing, and constant functions. We also examine what is called the difference quotient. These topics are subjects the student will explore with great detail in calculus.

Definition 1. Increasing Functions, Decreasing Functions, Constant Functions. Let $f$ be a function and $(a, b)$ be some interval in the domain of $f$. The function is called

- **increasing** over $(a, b)$ if $f(x) < f(y)$ for every $x < y$,
- **decreasing** over $(a, b)$ if $f(x) > f(y)$ for every $x < y$, and
- **constant** over $(a, b)$ if $f(x) = f(y)$ for every $x$ and $y$

(where $a < x < y < b$).

Example 1.a. Determine over which intervals the function is increasing, decreasing, or constant.

![Graph of a function](image)

Increasing: $(-5, -2) \cup (1, 4)$
Decreasing: $(-1, 1) \cup (4, 5)$
Constant: $(-2, -1)$

Example 1.b. Determine over which intervals the function is increasing, decreasing, or constant.
Increasing: $(-2, 0)$
Decreasing: $(-3, -2) \cup (0, 2)$
Constant: $(-5, -3) \cup (2, 5)$

**Example 1.c.** Determine over which intervals the function is increasing, decreasing, or constant.

One of the topics a student studies in calculus is how a function changes. For example, a calculus student might be interested in the place a function changes from increasing to decreasing. Another topic the calculus student will study is called the difference quotient.

**Definition 2. Difference Quotient.** For a function $f(x)$ and an increment $h$, the difference quotient is

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$
Example 2.a. Let \( f(x) = 2x + 3 \). Find the difference quotient.

\[
\frac{f(x+h) - f(x)}{h} = \frac{[2(x+h) + 3] - (2x + 3)}{h} \quad (h \neq 0)
\]

\[
= \frac{2x + 2h + 3 - 2x - 3}{h}
\]

\[
= \frac{2h}{h}
\]

\[
= 2
\]

Example 2.b. Let \( f(x) = 5x - 6 \). Find the difference quotient.

\[
\frac{f(x+h) - f(x)}{h} = \frac{[5(x+h) - 6] - (5x - 6)}{h} \quad (h \neq 0)
\]

\[
= \frac{5x + 5h - 6 - 5x + 6}{h}
\]

\[
= \frac{5h}{h}
\]

\[
= 5
\]

Example 2.c. Let \( f(x) = x^2 + 1 \). Find the difference quotient.

\[
\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h} \quad (h \neq 0)
\]

\[
= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h}
\]

\[
= \frac{2xh + h^2}{h}
\]

\[
= \frac{h(2x + h)}{h}
\]

\[
= 2x + h
\]

Example 2.d. Let \( f(x) = x^2 - 4x \). Find the difference quotient.

38
\[
\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^2 - 4(x + h)] - (x^2 - 4x)}{h} \quad (h \neq 0)
\]
\[
= \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h}
\]
\[
= \frac{2xh + h^2 - 4h}{h}
\]
\[
= \frac{h(2x + h - 4)}{h}
\]
\[
= 2x + h - 4
\]
2.3 Symmetry of Functions

In the study of functions, there are two important symmetries we consider.

Definition 1. Even Function. A function $f$ is called *even* if

$$f(-x) = f(x).$$

Definition 2. Odd Function. A function $f$ is called *odd* if

$$f(-x) = -f(x).$$

Example 1.a. Determine if $f$ is even, odd, or neither.

$$f(x) = x^2 - 4$$
$$f(-x) = (-x)^2 - 4$$
$$= x^2 - 4$$
$$= f(x)$$

Thus, $f$ is an even function. Observe the graph of $f$ below.

Notice that an even function is symmetric with respect to the $y$-axis.

Example 1.b. Determine if $g$ is even, odd, or neither.

$$g(x) = x^3 - 2x$$
$$g(-x) = (-x)^3 - 2(-x)$$
$$= -x^3 + 2x$$
$$= -(x^3 - 2x)$$
$$= -g(x)$$

Thus, $g$ is an odd function. Observe the graph of $g$ below.
Notice that an odd function is symmetric with respect to the origin.

**Example 1.c.** Determine if $h$ is even, odd, or neither.

\[
h(x) = (x - 2)^2
\]

\[
h(-x) = [(-x) - 2]^2
\]

\[
= x^2 + 4x + 4
\]

Note:

\[
h(x) = (x - 2)^2 = x^2 - 4x + 4
\]

\[
-h(x) = -(x - 2)^2 = -x^2 + 4x - 4
\]

Thus, $h$ is neither even nor odd. Observe the graph of $h$ below.

Notice the graph is neither symmetric about the $y$-axis or symmetric about the origin.
2.4 A List of Basic Functions

In this section, our goal is to introduce a number of basic functions. The intent of this introduction is to reinforce the idea that a function is a rule. These rules can operate in all sort of interesting ways, but in this section, we introduce some fairly basic rules.

**Definition 1.a. The Constant Function.** The function

\[ f(x) = c, \]

where \( c \) is a real number, is called the *constant function*.

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & c \\
-1 & c \\
0 & c \\
1 & c \\
2 & c \\
\end{array}
\]

The constant function \( f(x) = 2 \) is graphed below.

*Definition 1.b. The Identity Function.** The function

\[ f(x) = x \]

is called the *identity function*.

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & -2 \\
-1 & -1 \\
0 & 0 \\
1 & 1 \\
2 & 2 \\
\end{array}
\]
Definition 1.c. The Square Function. The function
\[ f(x) = x^2 \]
is called the square function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Definition 1.d. The Cube Function. The function
\[ f(x) = x^3 \]
is called the cube function.
Definition 1.e. The Square Root Function. The function

\[ f(x) = \sqrt{x} \]

is called the square root function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$4$</td>
<td>$2$</td>
</tr>
<tr>
<td>$9$</td>
<td>$3$</td>
</tr>
</tbody>
</table>
Definition 1.f. The Cube Root Function. The function

\[ f(x) = \sqrt[3]{x} \]

is called the cube root function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
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</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

Definition 1.g. The Absolute Value Function. The function

\[ f(x) = |x| \]

is called the absolute value function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
2.5 Transformations of Functions

Our goal in this section is to graph many more functions. In the previous section, we introduced the basic graphs. Now, we will learn that if we know the basic graph, we can graph any graph that is similar to the basic graph. This will allow us to graph many functions.

Example 1.a. Graph $f(x) = x^2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

![Graph of $f(x) = x^2$.]
Example 1.b. Graph $g(x) = -x^2$.

Note $g(x) = -f(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-4</td>
</tr>
</tbody>
</table>

If the function is multiplied by $-1$, the graph is reflected over the $x$-axis.
Example 2.a. Graph $f(x) = x^2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>4</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Example 2.b. Graph $g(x) = x^2 + 1$.

Note $g(x) = f(x) + 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Example 2.c. Graph $g(x) = x^2 - 3$.

Note $g(x) = f(x) - 3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 2.d. Graph $g(x) = (x - 2)^2$.

Note $g(x) = f(x - 2)$.

<table>
<thead>
<tr>
<th>$x - 2$</th>
<th>$f(x - 2)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
Example 2.e. Graph $g(x) = (x + 3)^2$.

Note $g(x) = f(x + 3)$.

<table>
<thead>
<tr>
<th>$x + 3$</th>
<th>$f(x - 2)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>-5</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-4</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-1</td>
<td>4</td>
</tr>
</tbody>
</table>

In example 2, we noticed that adding or subtracting a constant resulted in a vertical shift. However, adding or subtracting inside of the square resulted in a horizontal shift. Example 3 combines the transformations we have observed.
Example 3.a. Graph $h(x) = (x + 4)^2 + 1$.

Let $f(x) = x^2$ and $g(x) = (x + 4)^2$.

Note $h(x) = g(x) + 1 = f(x + 4) + 1$.

<table>
<thead>
<tr>
<th>$x + 4$</th>
<th>$f(x + 4)$</th>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>-6</td>
<td>4</td>
<td>-6</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-5</td>
<td>1</td>
<td>-5</td>
<td>2</td>
</tr>
<tr>
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<td>0</td>
<td>-4</td>
<td>0</td>
<td>-4</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>1</td>
<td>-3</td>
<td>2</td>
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<tr>
<td>2</td>
<td>4</td>
<td>-2</td>
<td>4</td>
<td>-2</td>
<td>5</td>
</tr>
</tbody>
</table>
Example 3.b. Graph \( h(x) = (x - 5)^2 + 3 \).

Let \( f(x) = x^2 \) and \( g(x) = (x - 5)^2 \).

Note \( h(x) = g(x) + 3 = f(x - 5) + 3 \).

<table>
<thead>
<tr>
<th>( x - 5 )</th>
<th>( f(x - 5) )</th>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4</td>
<td>3</td>
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<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>
Example 3.c. Graph $k(x) = -(x - 3)^2 - 1$.

Let $f(x) = x^2$, $g(x) = (x - 3)^2$, and $h(x) = -(x - 3)^2$.

Note $k(x) = h(x) - 1 = -g(x) - 1 = -f(x - 3) - 1$.

<table>
<thead>
<tr>
<th>$x - 3$</th>
<th>$f(x - 3)$</th>
<th>$g(x)$</th>
<th>$h(x)$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>1</td>
<td>-4</td>
<td>-5</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>-4</td>
<td>-5</td>
</tr>
</tbody>
</table>

So far we have made transformations of our basic graph $f(x) = x^2$. We will notice all the transformations we have discovered apply to all of our basic graphs.
Example 4.a. Graph $f(x) = \sqrt{x}$.

\[
\begin{array}{c|c}
 x & f(x) \\
 0 & 0 \\
 1 & 1 \\
 4 & 2 \\
 9 & 3 \\
\end{array}
\]

Example 4.b. Graph $g(x) = \sqrt{x} - 2$.

Note $g(x) = f(x) - 2$.

\[
\begin{array}{c|c|c}
 x & f(x) & g(x) \\
 0 & 0 & -2 \\
 1 & 1 & -1 \\
 4 & 2 & 0 \\
 9 & 3 & 1 \\
\end{array}
\]
Example 4.c. Graph \( f(x) = \sqrt{x - 2} \).

Note \( g(x) = f(x - 2) \).

<table>
<thead>
<tr>
<th>( x - 2 )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>
Example 4.d. Graph $k(x) = -\sqrt{x+1} + 2$.

Let $f(x) = \sqrt{x}$, $g(x) = \sqrt{x+1}$, and $h(x) = -\sqrt{x+1}$.

Note $k(x) = h(x) + 2 = -g(x) + 2 = -f(x+1) + 2$.

<table>
<thead>
<tr>
<th>$x + 1$</th>
<th>$f(x + 1)$</th>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>-2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>8</td>
<td>-3</td>
<td>8</td>
<td>-1</td>
</tr>
</tbody>
</table>
Example 5.a. Graph $f(x) = x^3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>
Example 5.b. Graph \( h(x) = (x - 5)^3 - 2 \).

Let \( f(x) = x^3 \) and \( g(x) = (x - 5)^3 \).

Note \( h(x) = g(x) - 2 = f(x - 5) - 2 \).

\[
\begin{array}{c|c|c|c|c|c}
 x - 5 & f(x - 5) & x & g(x) & x & h(x) \\
-2 & -8 & 3 & -8 & 3 & -10 \\
-1 & -1 & 4 & -1 & 4 & -3 \\
0 & 0 & 5 & 0 & 5 & -2 \\
1 & 1 & 6 & 1 & 6 & -1 \\
2 & 8 & 7 & 8 & 7 & 6 \\
\end{array}
\]
Example 6.a. Graph $f(x) = |x|$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Example 6.b. Graph $k(x) = -|x + 2| + 1$.

Let $f(x) = |x|$, $g(x) = |x + 2|$, and $h(x) = -|x + 2|$.

Note $k(x) = h(x) + 1 = -g(x) + 1 = -f(x + 2) + 1$.

<table>
<thead>
<tr>
<th>$x + 2$</th>
<th>$f(x - 3)$</th>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
<td>-4</td>
<td>2</td>
<td>-4</td>
<td>-2</td>
<td>-4</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-3</td>
<td>1</td>
<td>-3</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
Example 7. The graph of the function $f$ is given below.

Graph the following functions:

(a) $g(x) = f(x) - 2$

(b) $h(x) = f(x + 2)$
(c) \( k(x) = -f(x - 1) + 2 \)
2.6 Operations on Functions

In this section, we will study operations on functions. The math student is familiar with operations on numbers: addition, subtract, multiplication, and division. All of these operations can be performed on functions. Special care must be observed during division, but for the most part, these basic operations on functions are fairly intuitive.

**Definition 1. Basic Operations on Functions.** Let \( f(x) \) and \( g(x) \) be functions. The following basic operations of addition, subtraction, multiplication, and division may be performed on the functions as follows:

- \((f + g)(x) = f(x) + g(x)\)
- \((f - g)(x) = f(x) - g(x)\)
- \((f \cdot g)(x) = f(x) \cdot g(x)\)
- \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\)

If the domain of \( f(x) \) is \( A \) and the domain of \( g(x) \) is \( B \), then the domain of \( f + g, f - g, \text{ and } f \cdot g \) is \( A \cap B \). The domain of \( f/g \) is \((A \cap B) \setminus \{x \in B \mid g(x) = 0\}\).\(^1\)

**Example 1.** Consider the functions \( f(x) = 3x - 2 \) and \( g(x) = x + 7 \). Find \( f + g, f - g, f \cdot g, \text{ and } f/g \). State the domain of each function.

\[
(f + g)(x) = f(x) + g(x) \\
= (3x - 2) + (x + 7) \\
= 4x + 5
\]

\[
(f - g)(x) = f(x) - g(x) \\
= (3x - 2) - (x + 7) \\
= 3x - 2 - x - 7 \\
= 2x - 9
\]

\[
(f \cdot g)(x) = f(x) \cdot g(x) \\
= (3x - 2)(x + 7) \\
= 3x^2 + 19x - 14
\]

\(^1\)Let me explain this in class.
Since the domain of both \( f \) and \( g \) is all real number, the domains of \( f + g \), \( f - g \) and \( f \cdot g \) is also all real numbers. The domain of \( f/g \) is \((-\infty, -7) \cup (-7, \infty)\).

Besides the basic operations of addition, subtraction, multiplication, and division, there is another very important type of operation on functions. This operation is called composition.

**Definition 2. Composition of Functions.** Let \( f(x) \) and \( g(x) \) be functions. The composition of \( f \) and \( g \), denoted \( f \circ g \), is given by

\[
(f \circ g)(x) = f(g(x)).
\]

**Example 2.** Consider the functions \( f(x) = 3x - 2 \) and \( g(x) = x + 7 \). Find \( f \circ g \) and \( g \circ f \). State the domain of each function.

\[
(f \circ g)(x) = f(g(x))
\]

\[
= f(x + 7)
\]

\[
= 3(x + 7) - 2
\]

\[
= 3x + 21 - 2
\]

\[
= 3x + 19
\]

\[
(g \circ f)(x) = g(f(x))
\]

\[
= g(3x - 2)
\]

\[
= (3x - 2) + 7
\]

\[
= 3x + 5
\]

Since the domain of both \( f \) and \( g \) is all real number, the domain of \( f \circ g \) and \( g \circ f \) is also all real numbers.
2.7 Inverse Functions

Recall the definition of a function from earlier in the chapter. A function is a relation in which each element in the domain corresponds to exactly one element in the range. A special type of function is introduced below.

**Definition 1.** A function \( f \) is called **one-to-one** if each element in the range corresponds to exactly one element in the domain.

If a function is one-to-one, then it has an inverse function.

**Definition 2.** Two functions \( f \) and \( g \) are called inverse functions if

\[
(f \circ g)(x) = (g \circ f)(x) = x.
\]

It is customary to write the \( f^{-1} \) for the inverse function of \( f \).
3 Polynomial Functions

3.1 Quadratic Functions

Our goal in this section is to graph quadratic functions. We will begin by remembering the graph of the basic quadratic function \( f(x) = x^2 \), and the transformations to our basic graph. After recalling these basic ideas, we will develop a method to graph any quadratic function.

**Definition 1.** The vertex of a parabola is the point where the parabola achieves its minimum or maximum value.

**Example 1.a.** Graph \( f(x) = x^2 \).

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -2 & 4 \\
  -1 & 1 \\
  0 & 0 \\
  1 & 1 \\
  2 & 4 \\
\end{array}
\]

**Example 1.b.** Graph \( f(x) = (x + 4)^2 - 1 \).

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -6 & 4 \\
  -5 & 1 \\
  -4 & 0 \\
  -3 & 1 \\
  -2 & 4 \\
\end{array}
\]

**Example 1.c.** Graph \( f(x) = 2(x - 5)^2 + 3 \).

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  3 & 11 \\
  4 & 5 \\
  5 & 3 \\
  6 & 5 \\
  7 & 11 \\
\end{array}
\]

**Example 1.d.** Graph \( f(x) = -3(x - 3)^2 - 1 \).

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  1 & -13 \\
  2 & -4 \\
  3 & 1 \\
  4 & -4 \\
  5 & -13 \\
\end{array}
\]

Let’s revisit example 1.b. We were asked to graph \( f(x) = (x + 4)^2 - 1 \). We knew that we should shift the parabola 4 units to the left and one unit down – moving our vertex to \((-4, -1)\). What if we were asked to graph the following quadratic equation?

\[
f(x) = x^2 + 8x + 15
\]
The graph for this function is exactly the graph of Example 1.b. Notice:

\[ f(x) = (x + 4)^2 - 1 \]

\[ = x^2 + 8x + 16 - 1 \]

\[ = x^2 + 8x + 15 \]

Clearly, we will have better luck graphing this function if we write it as we did in Example 3.a. as opposed to how it is written above. Let’s state this precisely...

**Definition 2. General Form of a Parabola.** A quadratic function is said to be in *general form* if it is written as

\[ f(x) = ax^2 + bx + c. \]

**Definition 3. Standard Form of a Parabola.** A quadratic function is said to be in *standard form* if it is written as

\[ f(x) = a(x - h)^2 + k. \]

The benefit of writing the quadratic function in standard form is we know the location of the vertex is given by \((h,k)\): observe the basic graph \(f(x) = x^2\) is translated \(h\) units to the right and \(k\) units up. The next example will show us the technique to convert a quadratic function from general form to standard form. This technique is called *completing the square*.

**Example 2.a.** Graph \(f(x) = x^2 + 6x + 7\).

\[ f(x) = x^2 + 6x + 7 \]

\[ = (x^2 + 6x + 9) + 7 - 9 \]

\[ = (x + 3)^2 - 2 \]

**Example 2.b.** Graph \(f(x) = x^2 - 8x + 19\).

\[ f(x) = x^2 - 8x + 19 \]

\[ = (x^2 - 8x + 16) + 19 - 16 \]

\[ = (x - 4)^2 + 3 \]
Example 2.c. Graph $f(x) = 2x^2 - 4x - 3$.

\[
\begin{align*}
f(x) &= 2x^2 - 4x - 3 \\
&= (2x^2 - 4x ) - 3 \\
&= 2(x^2 - 2x ) - 3 \\
&= 2(x^2 - 2x + 1) - 3 - 2 \\
&= 2(x - 1)^2 - 5
\end{align*}
\]

Example 2.d. Graph $f(x) = x^2 - 5x + 1$.

\[
\begin{align*}
f(x) &= x^2 - 5x + 1 \\
&= (x^2 - 5x ) + 1 \\
&= \left(x^2 - 5x + \frac{25}{4}\right) + 1 - \frac{25}{4} \\
&= \left(x - \frac{5}{2}\right)^2 - \frac{21}{4}
\end{align*}
\]

We should notice at this point it might be reasonable to conclude that we could use this technique for any quadratic function. This result, if true, could be more powerful than one might realize. Example 3 shows this conclusion is in fact true.

Example 3. Write the quadratic function $f(x) = ax^2 + bx + c$ in standard form.

\[
\begin{align*}
f(x) &= ax^2 + bx + c \\
&= (ax^2 + bx ) + c \\
&= a\left(x^2 + \frac{b}{a}x \right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}
\end{align*}
\]

From the previous examples, we know the transformations for this quadratic function, but this quadratic function is really any quadratic function. Any quadratic function written in general form is shifted $-\frac{b}{2a}$ units right and $\frac{4ac - b^2}{4a}$ units up. Stated easier, the vertex for any quadratic function is located at

\[
\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).
\]
Usually this formula is written in another form.

**Example 4.** Evaluate the quadratic function $f(x) = ax^2 + bx + c$ for $x = -\frac{b}{2a}$.

$$f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$

$$= a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right) + c$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$= \frac{b^2 - 2b^2}{4a} + \frac{4ac}{4a}$$

$$= \frac{4ac - b^2}{4a}$$

The points on the graph of a function are always given by $(x, f(x))$, so clearly the result arrived to in Example 4 was expected. We point it out in any case to notice that we may simply state the vertex of any quadratic function $f(x) = ax^2 + bx + c$ is located at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

**Example 5.a.** Graph $f(x) = 3x^2 - 6x + 2$.

We may graph this quadratic function by first finding the vertex.

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$= \left(-\frac{(-6)}{2(3)}, f\left(-\frac{b}{2a}\right)\right)$$

$$= \left(1, f(1)\right)$$

Evaluating the function at $x = 1$, we have

$$f(1) = 3(1)^2 - 6(1) + 2 = -1.$$  

Thus, the vertex of the parabola is located at $(1, -1)$.

**Example 5.b.** Graph $f(x) = -2x^2 - 7x + 5$. 

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We may graph this quadratic function by first finding the vertex.

\[
\left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right)
\]

\[
\left( -\frac{-7}{2(-2)}, f\left( -\frac{b}{2a} \right) \right)
\]

\[
\left( -\frac{7}{4}, f\left( -\frac{7}{4} \right) \right)
\]

Evaluating the function at \( x = -\frac{7}{4} \), we have

\[
f\left( -\frac{7}{4} \right) = -2\left( -\frac{7}{4} \right)^2 - 7\left( -\frac{7}{4} \right) + 5 = \frac{89}{8}.
\]

Thus, the vertex of the parabola is located at \( \left( -\frac{7}{4}, \frac{89}{8} \right) \).
3.2 Polynomial Functions

This chapter is almost entirely devoted to polynomial functions.

**Definition 1. Polynomial Function.** A polynomial function is a function of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0, \]

where \( n \) is a nonnegative integer and each \( a_i \) is a real number. Assuming \( a_n \neq 0 \), the degree of the polynomial function is \( n \) and \( a_n \) is called the leading coefficient.

The reader should be well familiar now with polynomial functions of degree 0, 1, or 2:

- We recognize a polynomial function of degree 0 to be the constant function \( f(x) = a_0 \).
- We recognize a polynomial function of degree 1 to be a linear function \( f(x) = a_1 x + a_0 \).
- We recognize a polynomial function of degree 2 to be a quadratic function \( f(x) = a_2 x^2 + a_1 x + a_0 \).

Thus, our focus is mainly on polynomial functions of degree 3 or higher. While we were able to fully describe any 2nd degree polynomial function in the previous section, higher degree polynomials are not as easily described. We will begin to develop an understanding of these functions, but the student should continue on to calculus to see the full development of the description of these polynomial functions.

Let us begin this study by examining the basic power functions.

**Definition 2. Power Function.** The polynomial function \( f(x) = x^n \) is called a power function.

**Example 1.** Graph the following power functions.

a. \( f(x) = x^1 \)

b. \( f(x) = x^2 \)

c. \( f(x) = x^3 \)

d. \( f(x) = x^4 \)

e. \( f(x) = x^5 \)

f. \( f(x) = x^6 \)

g. \( f(x) = x^7 \)

We should notice all of the odd (or even) power functions behave in a similar way. In particular, we want to make note of the end-behavior of each of the power functions.

**Definition 3. End-behavior.** The end-behavior of a function is the value \( f(x) \) approaches as \( x \) approaches \(-\infty\) or as \( x \) approaches \( \infty \).

**Example 2.** Classify the end-behavior of each of the power functions in Example 1.

a. For \( f(x) = x \), as \( x \) approaches \(-\infty \), \( f(x) \) approaches \(-\infty \). As \( x \) approaches \( \infty \), \( f(x) \) approaches \( \infty \). We will use the following notation to describe this behavior. As \( x \to -\infty \), \( f(x) \to -\infty \). As \( x \to \infty \), \( f(x) \to \infty \).

b. For \( f(x) = x^2 \), as \( x \to -\infty \), \( f(x) \to \infty \). As \( x \to \infty \), \( f(x) \to \infty \).
c. For \( f(x) = x^3 \), as \( x \to -\infty \), \( f(x) \to -\infty \). As \( x \to \infty \), \( f(x) \to \infty \).
d. For \( f(x) = x^4 \), as \( x \to -\infty \), \( f(x) \to \infty \). As \( x \to \infty \), \( f(x) \to \infty \).
e. For \( f(x) = x^5 \), as \( x \to -\infty \), \( f(x) \to -\infty \). As \( x \to \infty \), \( f(x) \to \infty \).
f. For \( f(x) = x^6 \), as \( x \to -\infty \), \( f(x) \to \infty \). As \( x \to \infty \), \( f(x) \to \infty \).
g. For \( f(x) = x^7 \), as \( x \to -\infty \), \( f(x) \to -\infty \). As \( x \to \infty \), \( f(x) \to \infty \).

Example 3. Graph the following power functions. What happened to the end-behavior?

a. \( f(x) = -x \)
b. \( f(x) = -x^2 \)
c. \( f(x) = -x^3 \)
d. \( f(x) = -x^4 \)
e. \( f(x) = -x^5 \)

Example 4. Observe the graphs the following polynomial functions. State the end-behavior of each.

a. \( f(x) = x^4 + x^3 \)
b. \( f(x) = x^4 + 10x^3 \)
c. \( f(x) = x^4 + 100x^3 \)
d. \( f(x) = x^5 + x^4 \)
e. \( f(x) = -x^5 + x^4 \)

Observation 1. The end-behavior of any polynomial function is the same as the end-behavior of its highest degree term.

Definition 4. Zero of a Function. If \( f(c) = 0 \), then \( c \) is called a zero of the function.

Observation 2. If \( c \) is a zero of a function, then \((c,0)\) is an \( x\)-intercept on the graph of the function.

Besides understanding end-behavior and \( x\)-intercepts, the other concept we will develop follows from our work solving nonlinear inequalities. Consider the following example.

Example 5. Sketch a graph of \( f(x) = (x + 4)(x + 1)(x - 2) \).

First, we may observe the following \( x\)-intercepts: \((-4,0), (-1,0), \) and \((2,0)\).

Since \( f(x) = (x + 4)(x + 1)(x - 2) = x^3 + 3x^2 - 6x - 8 \), we know the end-behavior is the same as \( x^3 \): as \( x \to -\infty \), \( f(x) \to -\infty \), and as \( x \to \infty \), \( f(x) \to \infty \).

But what happens for \( x \) values between \(-4\) and \(2\)? Recall how we found the solution to \((x + 4)(x + 1)(x - 2) > 0\). We will examine 2 cases:

- Case (1): If \( x \) is in the interval \((-4,-1)\). Then:
  - \( (x + 4) \) is positive,
  - \( (x + 1) \) is negative, and
  - \( (x - 2) \) is negative.

Thus, \((x + 4)(x + 1)(x - 2)\) is positive, so \( f(x) > 0 \).
• Case (2): If \( x \) is in the interval \((-1, 2)\). Then:
  
  ◦ \((x + 4)\) is positive,
  ◦ \((x + 1)\) is positive, and
  ◦ \((x - 2)\) is negative.

  Thus, \((x + 4)(x + 1)(x - 2)\) is negative, so \(f(x) < 0\).

Finally, we may sketch the graph.

Let us explore a few similar polynomial functions in the next examples.

**Example 6.** Sketch a graph of \(f(x) = (x + 4)(x + 1)^2(x - 2)\).

Notice we have the same zeros of the function as in Example 5, so we have the same \(x\)-intercepts.

The end-behavior is different. Observe

\[
f(x) = (x + 4)(x + 1)^2(x - 2) = x^4 + 4x^3 - 3x^2 - 14x - 8,
\]

so the end-behavior is the same as \(x^4\): as \(x \to -\infty\), \(f(x) \to \infty\), and as \(x \to \infty\), \(f(x) \to \infty\).

Consider the same 2 cases:

- **Case (1):** If \( x \) is in the interval \((-4, -1)\). Then:
  
  ◦ \((x + 4)\) is positive,
  ◦ \((x + 1)\) is negative, and
  ◦ \((x - 2)\) is negative.

  Thus, \((x + 4)(x + 1)^2(x - 2)\) is negative, so \(f(x) < 0\).

- **Case (2):** If \( x \) is in the interval \((-1, 2)\). Then:

  ◦ \((x + 4)\) is positive,
  ◦ \((x + 1)\) is positive, and
  ◦ \((x - 2)\) is negative.

  Thus, \((x + 4)(x + 1)^2(x - 2)\) is negative, so \(f(x) < 0\).

Finally, we may sketch the graph.

**Example 7.** Sketch a graph of \(f(x) = (x + 4)^2(x + 1)^3(x - 2)\).

Notice we have the same zeros of the function as in Examples 5 and 6, so we have the same \(x\)-intercepts.

Observe

\[
f(x) = (x + 4)^2(x + 1)^3(x - 2) = x^6 + \ldots + 32,
\]

so the end-behavior is the same as \(x^6\): as \(x \to -\infty\), \(f(x) \to \infty\), and as \(x \to \infty\), \(f(x) \to \infty\).

Consider the same 2 cases:
• Case (1): If \( x \) is in the interval \((-4, -1)\). Then:
  - \((x + 4)\) is positive,
  - \((x + 1)\) is negative, and
  - \((x - 2)\) is negative.

  Thus, \((x + 4)^2(x + 1)^3(x - 2)\) is positive, so \(f(x) > 0\).

• Case (2): If \( x \) is in the interval \((-1, 2)\). Then:
  - \((x + 4)\) is positive,
  - \((x + 1)\) is positive, and
  - \((x - 2)\) is negative.

  Thus, \((x + 4)^2(x + 1)^3(x - 2)\) is negative, so \(f(x) < 0\).

Finally, we may sketch the graph.

**Definition 5. Multiplicity.** If \((x - c)^n\) is a factor of \(f(x)\), but \((x - c)^{n+1}\) is not a factor of \(f(x)\), then \(c\) is a zero of multiplicity \(n\).

**Observation 3.** If \(c\) is a zero of multiplicity \(n\), then:
  - if \(n\) is odd, the graph crosses the \(x\)-axis,
  - if \(n\) is even, the graph touches the \(x\)-axis, but does not cross.

**Theorem 1. Intermediate Value Theorem.** If \(p(x)\) is a polynomial function and \(a\) and \(b\) are real numbers with \(a < b\), then if either
  - \(p(a) < 0 < p(b)\), or
  - \(p(b) < 0 < p(a)\),

then there exists a real number \(c\) such that \(a < c < b\) and \(f(c) = 0\).
3.3 The Division Algorithm

In this section we state the division algorithm and use it to prove an interesting result: the remainder theorem.

**Proposition 1. The Division Algorithm.** Let \( p(x) \) be a polynomial of degree \( m \) and let \( d(x) \) be a nonzero polynomial of degree \( n \) where \( m \geq n \). Then there exists unique polynomials \( q(x) \) and \( r(x) \) such that

\[
p(x) = d(x) \cdot q(x) + r(x)
\]

where the degree of \( q(x) \) is \( m - n \) and the degree of \( r(x) \) is less than \( n \).

The polynomial \( d(x) \) is called the divisor, \( q(x) \) is called the quotient, and \( r(x) \) is called the remainder.

**Theorem 1. The Remainder Theorem.** Let \( p(x) \) be a polynomial. Then \( p(c) = r(x) \) where \( r(x) \) is the remainder from the division algorithm with \( d(x) = x - c \).

Note that the degree of \( r(x) \) is 0 since the degree of \( d(x) \) is 1.

**Example 1.** Divide \( p(x) = x^2 - 4x + 7 \) by \( d(x) = x - 4 \). Then evaluate \( p(4) \).
3.4 The Factor Theorem

In the previous section, we used the division algorithm to arrive at an interesting result: the remainder theorem. In this section, we will use the division algorithm to arrive at another result: the factor theorem.

**Theorem 1. The Factor Theorem.** Let \( p(x) \) and \( d(x) \) be polynomials. If \( r(x) = 0 \) by the division algorithm, then \( d(x) \) is a factor of \( p(x) \).

**Theorem 2. The Rational Zero Theorem.** Let \( p(x) \) be a polynomial function with integer coefficients:

\[
p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0.
\]

Then any rational zero of the polynomial will be of the form

\[
\pm \frac{\text{factor of } a_0}{\text{factor of } a_n}
\]

leading coefficient \( a_n \) and constant term \( a_0 \). If \( r(x) = 0 \) by the division algorithm, then \( d(x) \) is a factor of \( p(x) \).

**Proposition 1. Complex Conjugate Theorem.** Let \( p(x) \) be a polynomial with real coefficients. If \( a + bi \) is a zero of the polynomial, then its complex conjugate \( a - bi \) is also a zero of the polynomial.
4 Exponential and Logarithmic Functions

This entire chapter is perhaps the most important chapter for any student continuing on to calculus. This chapter represents the culmination of the algebra student’s study of exponents. While the student should be familiar with the basic concepts of exponents, we will examine exponents in a completely different way than the student has seen previously.

4.1 Exponential Functions

Definition. Exponential Function. The function

\[ f(x) = b^x, \]

where \( b > 0 \) and \( b \neq 1 \), is called an exponential function.

In this first section, we simply want a basic understanding of how an exponential function behaves. In particular, we will note the domain and range of an exponential function and observe that all of our basic transformations hold. We also introduce the natural base, \( e \).

Example 1.a. Graph \( f(x) = 2^x \). State its domain and range.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>-1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Example 1.b. Graph \( f(x) = 2^x + 3 \). State its domain and range.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( \frac{13}{4} )</td>
</tr>
<tr>
<td>-1</td>
<td>( \frac{3}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

Example 1.c. Graph \( f(x) = 2^{x+3} \). State its domain and range.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>-4</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
</tbody>
</table>

Example 1.d. Graph \( f(x) = -2^x \). State its domain and range.
Example 1.e. Graph $f(x) = 2^{-x}$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>-1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
</tbody>
</table>

Example 2.a. Graph $f(x) = \left(\frac{1}{2}\right)^x$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

Example 2.b. Graph $f(x) = -\left(\frac{1}{2}\right)^{x-4} + 1$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

Example 3.a. Graph $f(x) = 3^x$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$\frac{1}{9}$</td>
</tr>
<tr>
<td>-1</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
</tbody>
</table>

Example 3.b. Graph $f(x) = 3^{-x}$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>9</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{9}$</td>
</tr>
</tbody>
</table>
Example 4.a. Graph $f(x) = 10^x$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1/10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Example 4.b. Graph $f(x) = 10^{2-x} - 4$. State its domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>-39/10</td>
</tr>
</tbody>
</table>

We have seen the base of the exponential function be 2, $\frac{1}{2}$, 3, and 10. The base that comes up most often in real world applications, however, is called the natural base, $e$. Before defining the natural base, we will point out that there are a couple definitions we could give for this number. In fact, we will have to give a definition for this number, because the natural base is an irrational number like $\pi$. The number $\pi$ may be approximated as 3.14, but $\pi \neq 3.14$. Instead $\pi = 3.14159 \ldots$. Notice this is not a very good definition of the number $\pi$. The definition of $\pi$ is $\pi$ is the ratio of the circumference of a circle to its diameter, or $\pi = \frac{C}{d}$. It just happens that $\frac{C}{d} = 3.14159 \ldots$ for any circle. As we mentioned above, there are a couple definitions we could give for $e$.

**Definition. Natural Base.** Consider the expression $(1 + \frac{1}{n})^n$ for various values of $n$. See the table below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$(1 + \frac{1}{n})^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>2.59374</td>
</tr>
<tr>
<td>100</td>
<td>2.70481</td>
</tr>
<tr>
<td>1,000</td>
<td>2.71692</td>
</tr>
<tr>
<td>10,000</td>
<td>2.71815</td>
</tr>
<tr>
<td>100,000</td>
<td>2.71827</td>
</tr>
</tbody>
</table>

As $n$ gets bigger, the expression $(1 + \frac{1}{n})^n$ gets bigger as well, but this sequence has an upper bound. This particular upper bound, called its limit, is called the natural base, $e$.

$$e = 2.71828\ldots$$

The reader might be wondering what is so special about this number. I highly encourage the reader to explore this question further by signing up for Calculus I in the future.

Example 5.a. Graph $f(x) = e^x$. State its domain and range.

---

2The definition given for the natural base in calculus will be constructed in a different manner.
\begin{tabular}{|c|c|}
\hline
$x$ & $f(x)$ \\
\hline
-2 & $e^{-2} \approx 0.135$ \\
-1 & $e^{-1} \approx 0.368$ \\
0 & 1 \\
1 & $e \approx 2.718$ \\
2 & $e^2 \approx 7.389$ \\
\hline
\end{tabular}

**Example 5.b.** Graph $f(x) = e^{-x}$. State its domain and range.

\begin{tabular}{|c|c|}
\hline
$x$ & $f(x)$ \\
\hline
-2 & $e^2 \approx 7.389$ \\
-1 & $e \approx 2.718$ \\
0 & 1 \\
1 & $e^{-1} \approx 0.368$ \\
2 & $e^{-2} \approx 0.135$ \\
\hline
\end{tabular}

**Example 5.c.** Graph $f(x) = -3e^x + 1$. State its domain and range.

\begin{tabular}{|c|c|}
\hline
$x$ & $f(x)$ \\
\hline
-1 & $-3e^{-1} + 1 \approx -0.104$ \\
0 & -2 \\
1 & $-3e + 1 \approx -7.155$ \\
\hline
\end{tabular}

**Example 5.d.** Graph $f(x) = e^{2x} - 3$. State its domain and range.

\begin{tabular}{|c|c|}
\hline
$x$ & $f(x)$ \\
\hline
-0.5 & $e^{-1} - 3 \approx -2.632$ \\
0 & -2 \\
0.5 & $e - 3 \approx -0.282$ \\
\hline
\end{tabular}
4.2 Logarithmic Functions

Recall from Chapter 2 our study of inverse functions. If a function is one-to-one, then that function has an inverse function. For example, consider the function \( f(x) = 2x + 1 \). What if we wanted to know when the value of the function was equal to 6? We could set \( 2x + 1 = 6 \) and solve for \( x \) (\( x = \frac{5}{2} \)). Or if we wanted to know when the value of the function was \(-17\)? Again, set \( 2x + 1 = -17 \) and solve for \( x \) (\( x = -9 \)). But we developed a much nicer way to approach this problem. Instead of manipulating our function each time, we simply found its inverse function: \( f^{-1}(x) = \frac{x-1}{2} \). Note that \( f^{-1}(6) = \frac{5}{2} \) and \( f^{-1}(-17) = -9 \). Now let’s explore this same idea with exponential functions.

**Example 0.** Consider the exponential function \( f(x) = 2^x \). Find the value of the function such that:

(a) \( f(x) = 8 \)
(b) \( f(x) = \frac{1}{2} \)
(c) \( f(x) = 13 \)

By observation, we might be able to conclude that \( f(3) = 2^3 = 8 \) and \( f(-1) = 2^{-1} = \frac{1}{2} \). However, trying to solve for \( x \) in the equation

\[
\begin{align*}
  f(x) &= 2^x = 13
\end{align*}
\]

appears more difficult. One might be tempted to think there is no value of \( x \) that could be plugged in to make \( 2^x = 13 \), but simply playing around with a calculator should convince the student we could find some value of \( x \). Observe the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>3.25</td>
<td>9.5136...</td>
</tr>
<tr>
<td>3.5</td>
<td>11.3137...</td>
</tr>
<tr>
<td>3.75</td>
<td>13.4543...</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

We want to see here that we will have some \( x \) value between 3.5 and 3.75 such that \( 2^x = 13 \). More accurately, we believe there is some \( x \) value – and we would find it pretty crazy if somehow \( 2^x \) skipped over 13 when we tried all possible values for \( x \). We simply need an easy way to write this number.

**Definition 1.a.** The logarithm base \( b \) of \( x \), denoted \( \log_b x \) is the value of \( y \) such that \( b^y = x \).

Our new definition allows us to answer this question easier. What value of \( x \) will make \( 2^x = 13 \)? The answer is \( \log_2 13 \). What number is \( \log_2 13 \)? Well that is the number \( x \) that makes \( 2^x = 13 \). We can write previous definition more simply.

**Definition 1.b.** If \( b^y = x \), then \( \log_b x = y \).

Notice the value of a logarithm is the exponent. That is to say, a logarithm is an exponent. Example 0 could be written in the following manner.

**Example 0 (revisited).** Evaluate the following logarithms.

(a) \( \log_2 8 \)
Part (a) asks for what exponent $2^x = 8$. Thus, $\log_2 8 = 3$ (as $2^3 = 8$). Similarly, part (b) asks what exponent $2^y = \frac{1}{2}$. Thus, $\log_2 \frac{1}{2} = -1$ (as $2^{-1} = \frac{1}{2}$). For part (c), we cannot simplify $\log_2 13$ to any rational number. However, we will note that

$$2^{\log_2 13} = 13$$

as $\log_2 13$ is the exponent that makes $2^x = 13$.

**Example 1.** Write the following exponential equations as logarithmic equations.

(a) $2^4 = 16$
(b) $5^3 = 125$
(c) $81^{\frac{1}{2}} = 9$
(d) $4^{-3} = \frac{1}{64}$

**Solution.**

(a) $\log_2 16 = 4$
(b) $\log_5 125 = 3$
(c) $\log_{81} 9 = \frac{1}{2}$
(d) $\log_4 \frac{1}{64} = -3$

**Example 2.** Write the following logarithmic equations as exponential equations.

(a) $\log_{10} 1000 = 3$
(b) $\log_3 243 = 5$
(c) $\log_{27} 3 = \frac{1}{3}$
(d) $-3 = \log_2 \frac{1}{8}$

**Solution.**

(a) $10^3 = 1000$
(b) $3^5 = 243$
(c) $27^{\frac{1}{3}} = 3$
(d) $2^{-3} = \frac{1}{8}$
Example 3. Evaluate the following logarithms.
(a) $\log_5 25$
(b) $\log_{10} 10000$
(c) $\log_3 1$
(d) $\log_2 64$
(e) $\log_{36} \frac{1}{6}$

Solution.
(a) $\log_5 25 = 2$
(b) $\log_{10} 10000 = 4$
(c) $\log_3 1 = 0$
(d) $\log_2 64 = 6$
(e) $\log_{36} \frac{1}{6} = -\frac{1}{2}$

Examples 2 (a) and 3 (b) are not written in the correct manner really. The correct notation is shown below.

Notation. For logarithms base 10 and base $e$ we use the following notation.

$$\log x = \log_{10} x$$
$$\ln x = \log_e x$$

Let us return to where we began this discussion.

Example 4. Find the inverse function of $f(x) = 2^x$.

Solution. Recall the inverse function interchanges the $x$ and $y$ values.

$$f(x) = 2^x$$
$$y = 2^x$$
$$x = 2^y$$
$$\log_2 x = y$$
$$f^{-1}(x) = \log_2 x$$

Example 5.a. Graph the function $f(x) = 2^x$. State the domain and range.
Example 5.b. Graph the function $f(x) = \log_2 x$. State the domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Example 6.a. Graph the function $f(x) = \log_3 x$. State the domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Example 6.b. Graph the function $f(x) = \log x$. State the domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Example 6.c. Graph the function $f(x) = \ln x$. State the domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Example 7.a. Graph the function $f(x) = \log_2(x - 3)$. State the domain and range.

Example 7.b. Graph the function $f(x) = \log_2 x - 3$. State the domain and range.
4.3 Properties of Logarithms

In this section, we explore the basic properties of logarithms. These properties follow directly from the definition of a logarithm. Recall \( \log_b x = y \) if and only if \( b^y = x \).

**Theorem 1.** For any \( b > 0 \) and \( b \neq 0 \), the following are true:

1. \( \log_b 1 = 0 \)
2. \( \log_b b = 1 \)
3. \( \log_b b^x = x \)
4. \( b^{\log_b x} = x \)

**Proof.** Following from the definition of a logarithm, observe the following:

1. \( b^0 = 1 \)
2. \( b^1 = b \)
3. \( b^x = b^x \)
4. \( \log_b x = \log_b x \)

Notice for (3) we simply wrote the logarithmic equation as an exponential equation. For (4), we wrote the exponential equation as a logarithmic equation.

**Corollary 1.** The following are true:

1. \( \ln 1 = 0 \)
2. \( \ln e = 1 \)
3. \( \ln e^x = x \)
4. \( e^{\ln x} = x \)

Similarly,

1. \( \log 1 = 0 \)
2. \( \log 10 = 1 \)
3. \( \log 10^x = x \)
4. \( 10^{\log x} = x \)

**Theorem 2.** For any \( b > 0 \) and \( b \neq 0 \), the following are true:

1. \( \log_b MN = \log_b M + \log_b N \)
2. \( \log_b \frac{M}{N} = \log_b M - \log_b N \)
3. \( \log_b M^N = N \log_b x \)
Proof.

1. Let \( \log_b M = x \) and \( \log_b N = y \). Then \( b^x = M \) and \( b^y = N \). Then,

\[
M \cdot N = b^x \cdot b^y = b^{x+y}
\]

Rewriting in logarithmic form, we have

\[
\log_b MN = x + y = \log_b M + \log_b N
\]

2. Let \( \log_b M = x \) and \( \log_b N = y \). Then \( b^x = M \) and \( b^y = N \). Then,

\[
\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y}
\]

Rewriting in logarithmic form, we have

\[
\log_b \frac{M}{N} = x - y = \log_b M - \log_b N
\]

3. Let \( \log_b M = x \) and \( \log_b M^N = z \). Then \( b^x = M \) and \( b^z = M^N \). Notice

\[
b^z = (b^x)^N = b^{Nx}.
\]

Thus,

\[
z = Nx
\]

\[
\log_b M^N = N \log_b M.
\]

Theorem 3. For any \( a > 0 \) and \( a \neq 1 \),

\[
\log_b x = \frac{\log_a x}{\log_a b}.
\]

Proof. Let \( \log_b x = y \), \( \log_a x = z \), and \( \log_a b = w \). Then \( b^y = x \), \( a^z = x \), and \( a^w = b \). Observe

\[
b^y = x
\]

\[
(a^w)^y = a^z
\]

\[
a^{wy} = a^z.
\]

Thus,

\[
w y = z
\]

\[
y = \frac{z}{w}
\]

\[
\log_b x = \frac{\log_a x}{\log_a b}.
\]
4.4 Exponential and Logarithmic Equations

Proposition 1. For any $b > 0$ and $b \neq 1$, if $b^x = b^y$, then $x = y$.

Proposition 2. For any $b > 0$, $b \neq 1$, $x > 0$, and $y > 0$, if $\log_b x = \log_b y$, then $x = y$. 
5 Linear Algebra

This chapter includes a quick review of solving systems of linear equations. We will quickly progress into a nicer way to solve these systems using matrices.

5.1 Two Variable Systems of Equations

The first two sections of this chapter are intended to be review. Thus, I have only included some examples. For a more in-depth discussion of linear equations in two variables, please refer to my Intermediate Algebra Lecture Notes.

Example 1.a. Solve the following system of equations using substitution.

\[
\begin{align*}
  x - 2y & = 8 \\
  4x + y & = 5
\end{align*}
\]

Solution. To solve by substitution, we will take one of our equations and solve for one of our variables. Let’s take our first equation \( x - 2y = 8 \) and solve for \( x \).

\[
\begin{align*}
  x & = 2y + 8
\end{align*}
\]

Now, we may substitute \( x = 2y + 8 \) into our other equation \( 4x + y = 5 \).

\[
\begin{align*}
  4x + y & = 5 \\
  4(2y + 8) + y & = 5 \\
  8y + 32 + y & = 5 \\
  9y & = -27 \\
  y & = -3
\end{align*}
\]

Since \( y = -3 \), we may substitute that value back into any equation to solve for \( x \).

\[
\begin{align*}
  x & = 2y + 8 \\
  x & = 2(-3) + 8 \\
  x & = 2
\end{align*}
\]

Thus, \((2, -3)\) is the solution of this system of equations.

Example 1.b. Solve the following system of equations using substitution.

\[
\begin{align*}
  5x + 3y & = 6 \\
  4x + 2y & = 2
\end{align*}
\]

Solution. To solve by substitution, we will take one of our equations and solve for one of our variables. Let’s take our second equation \( 4x + 2y = 2 \) and solve for \( y \).

\[
\begin{align*}
  4x + 2y & = 2 \\
  2y & = -4x + 2 \\
  y & = -2x + 1
\end{align*}
\]

Now, we may substitute \( y = -2x + 1 \) into our other equation \( 5x + 3y = 6 \).

\[
\begin{align*}
  5x + 3y & = 6 \\
  5x + 3(-2x + 1) & = 6 \\
  5x - 6x + 3 & = 6 \\
  -x & = 3 \\
  x & = -3
\end{align*}
\]
Since \( x = -3 \), we may substitute that value back into any equation to solve for \( y \).

\[
\begin{align*}
y &= -2x + 1 \\
y &= -2(-3) + 1 \\
y &= 7
\end{align*}
\]

Thus, \((-3, 7)\) is the solution of this system of equations.

**Example 2.a.** Solve the following system of equations using elimination.

\[
\begin{align*}
x - 2y &= 8 \\
4x + y &= 5
\end{align*}
\]

**Solution.** To solve by elimination, we will eliminate one of our variables. Let’s eliminate \( y \).

\[
\begin{align*}
x - 2y &= 8 \\
4x + y &= 5
\end{align*}
\]

\[
\begin{align*}
x - 2y &= 8 \\
2(4x + y) &= (5)2
\end{align*}
\]

\[
\begin{align*}
x - 2y &= 8 \\
8x + 2y &= 10
\end{align*}
\]

\[
9x = 18
\]

\[
x = 2
\]

Since \( x = 2 \), we may substitute that value back into any equation to solve for \( y \).

\[
\begin{align*}
4x + y &= 5 \\
4(2) + y &= 5 \\
8 + y &= 5
\end{align*}
\]

\[
y = -3
\]

Thus, \((2, -3)\) is the solution of this system of equations.

**Example 2.b.** Solve the following system of equations using elimination.

\[
\begin{align*}
5x + 3y &= 6 \\
4x + 2y &= 2
\end{align*}
\]

**Solution.** To solve by elimination, we will eliminate one of our variables. Let’s eliminate \( y \).

\[
\begin{align*}
5x + 3y &= 6 \\
4x + 2y &= 2
\end{align*}
\]

\[
\begin{align*}
2(5x + 3y) &= (6)2 \\
-3(4x + 2y) &= (2)(-3)
\end{align*}
\]

\[
\begin{align*}
10x + 6y &= 12 \\
-12x - 6y &= -6
\end{align*}
\]

\[
-2x = 6
\]

\[
x = -3
\]
Since \( x = -3 \), we may substitute that value back into any equation to solve for \( y \).

\[
\begin{align*}
4x + 2y &= 2 \\
4(-3) + 2y &= 2 \\
-12 + 2y &= 2 \\
2y &= 14 \\
y &= 7
\end{align*}
\]

Thus, \((-3, 7)\) is the solution of this system of equations.

**Example 3.a.** Solve the following system of equations using elimination.

\[
\begin{align*}
4x + 6y &= 24 \\
y &= -\frac{2}{3}x + 4
\end{align*}
\]

**Solution.** First, let’s get rid of the fraction. Then we will move our variables over to the left side. Then we may choose a variable to eliminate.

\[
\begin{align*}
4x + 6y &= 24 \\
3(y) &= (-\frac{2}{3}x + 4) 3 \\
4x + 6y &= 24 \\
3y &= -2x + 12 \\
4x + 6y &= 24 \\
2x + 3y &= 12 \\
4x + 6y &= 24 \\
-2(2x + 3y) &= (12)(-2) \\
4x + 6y &= 24 \\
-4x - 6y &= -24 \\
0 &= 0
\end{align*}
\]

Arriving at this identity, we may conclude that this system is dependent. There are infinitely many solutions, or more specifically, the solution is any point on the line \( 4x + 6y = 24 \).

**Example 3.b.** Solve the following system of equations using elimination.

\[
\begin{align*}
y &= -3x + 4 \\
6x + 2y &= -2
\end{align*}
\]

**Solution.** First, we will move our variables over to the left side. Then we may choose a variable
to eliminate.

\[ y = -3x + 4 \]
\[ 6x + 2y = -2 \]

\[ 3x + y = 4 \]
\[ 6x + 2y = -2 \]

\[-2(3x + y) = (4)(-2)\]
\[ 6x + 2y = -2 \]

\[ 3x + y = 4 \]
\[ 6x + 2y = -2 \]

\[-2(3x + y) = (4)(-2)\]
\[ 6x + 2y = -2 \]

\[-6x - 2y = -8 \]
\[ 6x + 2y = -2 \]

\[ 0 = -10 \]

Arriving at this contradiction, we may conclude that this system is inconsistent. There is no solution.
5.2 Three Variable Systems of Equations

Example 1.a. Solve the following system of equations.

\[
\begin{align*}
  x + y + z &= 6 \\
  2x - 3y - z &= 5 \\
  3x + 2y - 2z &= -1
\end{align*}
\]

Solution. First, we will take two equations and eliminate a variable. Let’s take the first two equations and eliminate \(z\).

\[
\begin{align*}
  x + y + z &= 6 \\
  2x - 3y - z &= 5
\end{align*}
\]

\[
3x - 2y = 11
\]

Now, we want to take two different equations but eliminate the same variable: \(z\). We will take the first and last equations.

\[
\begin{align*}
  x + y + z &= 6 \\
  3x + 2y - 2z &= -1
\end{align*}
\]

\[
2(x + y + z) = (6)2 \\
3x + 2y - 2z = -1
\]

\[
2x + 2y + 2z = 12 \\
3x + 2y - 2z = -1
\]

\[
5x + 4y = 11
\]

We now have two equations with two variables. We may proceed as usual from here.

\[
\begin{align*}
  3x - 2y &= 11 \\
  5x + 4y &= 11
\end{align*}
\]

\[
2(3x - 2y) = (11)2 \\
5x + 4y = 11
\]

\[
6x - 4y = 22 \\
5x + 4y = 11
\]

\[
11x = 33 \\
x = 3
\]

We substitute \(x = 3\) into an equation containing only \(x\) and \(y\).

\[
\begin{align*}
  3x - 2y &= 11 \\
  3(3) - 2y &= 11 \\
  9 - 2y &= 11 \\
  -2y &= 2 \\
  y &= -1
\end{align*}
\]
Similarly, we substitute $x = 3$ and $y = -1$ into an equation containing $x$, $y$, and $z$.

\[
\begin{align*}
  x + y + z &= 6 \\
  (3) + (-1) + z &= 6 \\
  2 + z &= 6 \\
  z &= 4
\end{align*}
\]

Thus, $(3, -1, 4)$ is the solution to this system of equations.

**Example 1.b.** Solve the following system of equations.

\[
\begin{align*}
  3x + 4y &= -4 \\
  5y + 3z &= 1 \\
  2x - 5z &= 7
\end{align*}
\]

**Solution.** First, we will take two equations and eliminate a variable. Let’s take the last two equations and eliminate $z$.

\[
\begin{align*}
  5y + 3z &= 1 \\
  2x - 5z &= 7
\end{align*}
\]

\[
\begin{align*}
  5(5y + 3z) &= (1)5 \\
  3(2x - 5z) &= (7)3
\end{align*}
\]

\[
\begin{align*}
  25y + 15z &= 5 \\
  6x - 15z &= 21
\end{align*}
\]

\[
\begin{align*}
  6x + 25y &= 26
\end{align*}
\]

Notice we have two equations with two variables.

\[
\begin{align*}
  3x + 4y &= -4 \\
  6x + 25y &= 26
\end{align*}
\]

\[
\begin{align*}
  -2(3x + 4y) &= (-4)(-2) \\
  6x + 25y &= 26
\end{align*}
\]

\[
\begin{align*}
  -6x - 8y &= 8 \\
  6x + 25y &= 26
\end{align*}
\]

\[
\begin{align*}
  17y &= 34 \\
  y &= 2
\end{align*}
\]

We may now substitute $y = 2$ into an equation only containing $x$ and $y$ to solve for $x$ or into an
equation containing \(y\) and \(z\) to solve for \(z\).

\[
\begin{align*}
3x + 4y &= -4 \\
3x + 4(2) &= -4 \\
3x + 8 &= -4 \\
3x &= -12 \\
x &= -4 \\
\end{align*}
\]

\[
\begin{align*}
5y + 3z &= 1 \\
5(2) + 3z &= 1 \\
10 + 3z &= 1 \\
3z &= -9 \\
z &= -3
\end{align*}
\]

Thus, \((-4, 2, -3)\) is the solution to the system of equations.

**Example 1.c** Solve the following system of equations.

\[
\begin{align*}
2x + 3y - z &= 7 \\
-3x + 2y - 2z &= 7 \\
5x - 4y + 3z &= -10
\end{align*}
\]

**Solution.** First, we will take two equations and eliminate a variable. Let’s take the first two equations and eliminate \(z\).

\[
\begin{align*}
2x + 3y - z &= 7 \\
-3x + 2y - 2z &= 7 \\
-2(2x + 3y - z) &= (7)(-2) \\
-3x + 2y - 2z &= 7 \\
-4x - 6y + 2z &= -14 \\
-3x + 2y - 2z &= 7 \\
-7x - 4y &= -7
\end{align*}
\]

Now, we want to take two different equations but eliminate the same variable: \(z\). We will take the first and last equations.

\[
\begin{align*}
3(2x + 3y - z) &= (7)3 \\
5x - 4y + 3z &= -10 \\
6x + 9y - 3z &= 21 \\
5x - 4y + 3z &= -10 \\
11x + 5y &= 11
\end{align*}
\]
We now have two equations with two variables. We may proceed as usual from here.

\[
\begin{align*}
-7x - 4y &= -7 \\
11x + 5y &= 11
\end{align*}
\]

\[
\begin{align*}
5(-7x - 4y) &= (-7)5 \\
4(11x + 5y) &= (11)4
\end{align*}
\]

\[
\begin{align*}
-35x - 20y &= -35 \\
44x + 20y &= 44
\end{align*}
\]

\[
\begin{align*}
9x &= 9 \\
x &= 1
\end{align*}
\]

We substitute \( x = 1 \) into an equation containing only \( x \) and \( y \).

\[
\begin{align*}
-7x - 4y &= -7 \\
-7(1) - 4y &= -7 \\
-7 - 4y &= -7 \\
-4y &= 0 \\
y &= 0
\end{align*}
\]

Similarly, we substitute \( x = 1 \) and \( y = 0 \) into an equation containing \( x \), \( y \), and \( z \).

\[
\begin{align*}
2x + 3y - z &= 7 \\
2(1) + 3(0) - z &= 7 \\
2 - z &= 7 \\
-z &= 5 \\
z &= -5
\end{align*}
\]

Thus, \((1, 0, -5)\) is the solution to this system of equations.

**Example 2.a.** Solve the following system of equations.

\[
\begin{align*}
x + 2y - z &= 3 \\
2x + 3y - 5z &= 3 \\
5x + 8y - 11z &= 9
\end{align*}
\]

**Solution.** First, we will take two equations and eliminate a variable. Let’s take the first two equations and eliminate \( x \).

\[
\begin{align*}
x + 2y - z &= 3 \\
2x + 3y - 5z &= 3
\end{align*}
\]

\[
\begin{align*}
-2(x + 2y - z) &= (3)(-2) \\
2x + 3y - 5z &= 3
\end{align*}
\]

\[
\begin{align*}
-2x - 4y + 2z &= -6 \\
2x + 3y - 5z &= 3
\end{align*}
\]

\[
\begin{align*}
y - 3z &= -3
\end{align*}
\]
Now, we want to take two different equations but eliminate the same variable: \( x \). We will take the first and last equations.

\[
\begin{align*}
-5(x + 2y - z) &= (3)(-5) \\
5x + 8y - 11z &= 9
\end{align*}
\]

\[
\begin{align*}
-5x - 10y + 5z &= -15 \\
5x + 8y - 11z &= 9
\end{align*}
\]

\[
-2y - 6z = -6
\]

We now have two equations with two variables. We may proceed as usual from here.

\[
\begin{align*}
-y - 3z &= -3 \\
-2y - 6z &= -6
\end{align*}
\]

\[
\begin{align*}
-2(-y - 3z) &= (-3)2 \\
-2y - 6z &= -6
\end{align*}
\]

\[
\begin{align*}
2y + 6z &= -6 \\
-2y - 6z &= 6
\end{align*}
\]

\[0 = 0\]

Arriving at this identity, we conclude that this is a dependent system of equations, and thus it has infinitely many solutions. However, we are not saying any point \((x, y, z)\) will be a solution. We will describe the solution by introducing a parameter \(a\). Let \(z = a\). We will treat this parameter more as a constant than a variable. Observe our system.

\[
\begin{align*}
x + 2y - a &= 3 \\
2x + 3y - 5a &= 3 \\
5x + 8y - 11a &= 9
\end{align*}
\]

Examining the first two equations, we have:

\[
\begin{align*}
x + 2y - a &= 3 \\
2x + 3y - 5a &= 3
\end{align*}
\]

\[
\begin{align*}
x + 2y &= a + 3 \\
2x + 3y &= 5a + 3
\end{align*}
\]

\[
\begin{align*}
-2(x + 2y) &= -2(a + 3) \\
2x + 3y &= 5a + 3
\end{align*}
\]

\[
\begin{align*}
-2x - 4y &= -2a - 6 \\
2x + 3y &= 5a + 3
\end{align*}
\]

\[
\begin{align*}
-y &= 3a - 3 \\
y &= -3a + 3
\end{align*}
\]
Substituting $y = -3a + 3$ into our equation $x + 2y = a + 3$, we have:

$$x + 2(-3a + 3) = a + 3$$

$$x - 6a + 6 = a + 3$$

$$x = 7a - 3$$

We conclude $x = 7a - 3$, $y = -3a + 3$, and $z = a$. Thus, for any real number $a$, $(7a - 3, -3a + 3, a)$ is a solution to this system.

**Example 2.b.** Solve the following system of equations.

$$3x - y + 2z = 4$$
$$x - 5y + 4z = 3$$
$$6x - 2y + 4z = -8$$

**Solution.** First, we will take two equations and eliminate a variable. Let’s take the first and last equations and eliminate $x$.

$$3x - y + 2z = 4$$
$$6x - 2y + 4z = -8$$

$$-2(3x - y + 2z) = (4)(-2)$$
$$6x - 2y + 4z = -8$$

$$-6x + 2y - 4z = -8$$
$$6x - 2y + 4z = -8$$

$$0 = -16$$

Arriving at this contradiction, we conclude that this is an inconsistent system of equations, and thus it has **no solution**.
### 5.3 Matrices

**Definition 1.** An array is an arrange of objects.

**Definition 2.** An $m \times n$ matrix is an array of numbers with $m$ rows and $n$ columns.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Each $a_{i,j}$ is called an element, and $m \times n$ is called the order of the matrix.

**Proposition 1.** Two matrices $A$ and $B$ are equal if and only if they have the same order and each $a_{i,j} = b_{i,j}$.

**Definition 3.** Let $A$ and $B$ be $m \times n$ matrices. Then $A + B = X$ is an $m \times n$ matrix were each element

$$x_{i,j} = a_{i,j} + b_{i,j}.$$ 

Similarly, $A - B = X$ has elements $x_{i,j} = a_{i,j} - b_{i,j}$.

Note $A + B$ and $A - B$ are undefined if $A$ and $B$ have different orders.

**Example 1.** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $A + B$.

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

**Definition 4.** **Scalar Multiplication.** Let $A$ be an $m \times n$ matrix and let $k$ be a real number. Then $kA = X$ is an $m \times n$ matrix were each element

$$x_{i,j} = k \cdot a_{i,j}.$$ 

**Example 2.** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $3A$.

$$3A = 3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

**Example 3.** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $4A - B$.

$$4A - B = 4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 5 & 8 \end{bmatrix}$$

**Definition 5.** **Matrix Multiplication.** Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix. Then $AB = X$ is an $m \times p$ matrix were each element

$$x_{i,j} = a_{i,1} \cdot b_{1,j} + a_{i,2} \cdot b_{2,j} + \ldots + a_{i,n} \cdot b_{n,j}.$$
Example 4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $AB$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Example 5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $BA$.

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 6 \cdot 3 & 5 \cdot 2 + 6 \cdot 4 \\ 7 \cdot 1 + 8 \cdot 3 & 7 \cdot 2 + 7 \cdot 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Example 6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Find $CA$.

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \\ 5 \cdot 1 + 6 \cdot 3 & 5 \cdot 2 + 6 \cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \\ 23 & 34 \end{bmatrix}$$

Example 7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Find $AC$.

Note that $A$ is a $2 \times 2$ matrix and $B$ is a $3 \times 2$ matrix. Thus, $AC$ is undefined.
5.4 Gaussian Elimination

There are many applications for using matrices. One example we will explore is to solve systems of equations in a much nicer way than the student has previously done.

**Definition 1.** An *augmented matrix* is a matrix includes a line to separate the matrix into two parts.

We will use augmented matrices to represent systems of equations.

**Example 1.** A system of equations may be written as an augmented matrix as follows.

\[
\begin{align*}
x + y + z &= 6 \\
2x - 3y - z &= 5 \\
3x + 2y - 2z &= -1
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & | & 6 \\
2 & -3 & -1 & | & 5 \\
3 & 2 & -2 & | & -1
\end{bmatrix}
\]

**Proposition 1.** The following operations may be performed on an augmented matrix and preserve its associated system of equations:

1. Multiply each element of a row by a nonzero scalar.
2. Add a multiple of a row to another row.
3. Swap two rows.

These operations are called *elementary row operations*. The reader should verify for him or herself that these operations would not change the associated system of equations. We will adopt the notation shown in the following examples.

**Example 2.a.** Solve the following systems of equations by performing Gaussian elimination and back-substitution.

\[
\begin{align*}
x + y + z &= 6 \\
2x - 3y - z &= 5 \\
3x + 2y - 2z &= -1
\end{align*}
\]
\[
\begin{bmatrix}
1 & 1 & 1 & 6 \\
2 & -3 & -1 & 5 \\
3 & 2 & -2 & -1 \\
\end{bmatrix}
\]

\[-2R_1 + R_2 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & -5 & -3 & -7 \\
3 & 2 & -2 & -1 \\
\end{bmatrix}\]

\[-3R_1 + R_3 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & -5 & -3 & -7 \\
0 & -1 & -5 & -19 \\
\end{bmatrix}\]

\[R_2 \leftrightarrow R_3 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & -1 & -5 & -19 \\
0 & -5 & -3 & -7 \\
\end{bmatrix}\]

\[-R_2 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & 1 & 5 & 19 \\
0 & -5 & -3 & -7 \\
\end{bmatrix}\]

\[5R_2 + R_3 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & 1 & 5 & 19 \\
0 & 0 & 22 & 88 \\
\end{bmatrix}\]

\[\frac{1}{2}R_3 = \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & 1 & 5 & 19 \\
0 & 0 & 1 & 4 \\
\end{bmatrix}\]

Notice we are left with the system of equations below.

\[
x + y + z = 6 \\
y + 5z = 19 \\
z = 4
\]

From here, we may simply perform back-substitution to find our solution.

\[z = 4 \]

\[y + 5z = 19 \]
\[y + 20 = 19 \]
\[y = -1 \]

\[x + y + z = 6 \]
\[x - 1 + 4 = 6 \]
\[x + 3 = 6 \]
\[x = 3 \]

The solution to the system of equations is \((3, -1, 4)\).
Example 2.b. Solve the following systems of equations by performing Gaussian elimination and back-substitution.

\[
\begin{align*}
3x + 4y &= -4 \\
5y + 3z &= 1 \\
2x - 5z &= 7 \\
\end{align*}
\]

\[
\begin{bmatrix}
3 & 4 & 0 & | & -4 \\
0 & 5 & 3 & | & 1 \\
2 & 0 & -5 & | & 7 \\
\end{bmatrix}
\]

\[-R_3 + R_1 \Rightarrow \begin{bmatrix}
1 & 4 & 5 & | & -11 \\
0 & 5 & 3 & | & 1 \\
2 & 0 & -5 & | & 7 \\
\end{bmatrix}\]

\[-2R_1 + R_3 \Rightarrow \begin{bmatrix}
1 & 4 & 5 & | & -11 \\
0 & 5 & 3 & | & 1 \\
0 & -8 & -15 & | & 29 \\
\end{bmatrix}\]

\[\frac{1}{5}R_2 \Rightarrow \begin{bmatrix}
1 & 4 & 5 & | & -11 \\
0 & 1 & \frac{3}{5} & | & \frac{1}{5} \\
0 & -8 & -15 & | & 29 \\
\end{bmatrix}\]

\[8R_2 + R_3 \Rightarrow \begin{bmatrix}
1 & 4 & 5 & | & -11 \\
0 & 1 & \frac{3}{5} & | & \frac{1}{5} \\
0 & 0 & -\frac{51}{5} & | & \frac{153}{5} \\
\end{bmatrix}\]

\[-\frac{5}{3}R_3 \Rightarrow \begin{bmatrix}
1 & 4 & 5 & | & -11 \\
0 & 1 & \frac{3}{5} & | & \frac{1}{5} \\
0 & 0 & 1 & | & -3 \\
\end{bmatrix}\]

Notice we are left with the system of equations below.

\[
\begin{align*}
x + 4y + 5z &= -11 \\
y + \frac{3}{5}z &= \frac{1}{5} \\
z &= -3 \\
\end{align*}
\]

From here, we may simply perform back-substitution to find our solution.

\[
z = -3
\]

\[
y + \frac{3}{5}z = \frac{1}{5} \\
y = \frac{1}{5}
\]

\[
x + 4y + 5z = -11 \\
x + 8 - 15 = -11 \\
x - 7 = -11 \\
x = -4
\]

The solution to the system of equations is \((-4, 2, -3)\).
Example 2.c. Solve the following systems of equations by performing Gaussian elimination and back-substitution.

\[
\begin{align*}
2x + 3y - z &= 7 \\
-3x + 2y - 2z &= 7 \\
5x - 4y + 3z &= -10
\end{align*}
\]

\[
\begin{bmatrix}
2 & 3 & -1 & | & 7 \\
-3 & 2 & -2 & | & 7 \\
5 & -4 & 3 & | & -10
\end{bmatrix}
\]

\[R_1 \leftrightarrow R_2 \rightarrow
\begin{bmatrix}
-3 & 2 & -2 & | & 7 \\
2 & 3 & -1 & | & 7 \\
5 & -4 & 3 & | & -10
\end{bmatrix}
\]

\[2R_1 + R_3 \rightarrow
\begin{bmatrix}
1 & 8 & -4 & | & 21 \\
2 & 3 & -1 & | & 7 \\
5 & -4 & 3 & | & -10
\end{bmatrix}
\]

\[-2R_1 + R_2 \rightarrow
\begin{bmatrix}
1 & 8 & -4 & | & 21 \\
0 & -13 & 7 & | & -35 \\
5 & -4 & 3 & | & -10
\end{bmatrix}
\]

\[-5R_1 + R_3 \rightarrow
\begin{bmatrix}
1 & 8 & -4 & | & 21 \\
0 & -13 & 7 & | & -35 \\
0 & -44 & 23 & | & -115
\end{bmatrix}
\]

\[-\frac{1}{13}R_2 \rightarrow
\begin{bmatrix}
1 & 8 & -4 & | & \frac{21}{13} \\
0 & 1 & -\frac{7}{13} & | & \frac{35}{13} \\
0 & -44 & 23 & | & -115
\end{bmatrix}
\]

\[44R_2 + R_3 \rightarrow
\begin{bmatrix}
1 & 8 & -4 & | & \frac{21}{13} \\
0 & 1 & -\frac{7}{13} & | & \frac{35}{13} \\
0 & 0 & \frac{9}{13} & | & \frac{-5}{13}
\end{bmatrix}
\]

Notice we are left with the system of equations below.

\[
\begin{align*}
x + 8y - 4z &= 21 \\
y - \frac{7}{13}z &= \frac{35}{13} \\
z &= -5
\end{align*}
\]

From here, we may simply perform back-substitution to find our solution.

The solution to the system of equations is \((1, 0, -5)\).
Example 3.a. Solve the following systems of equations by performing Gaussian elimination and back-substitution.

\[
\begin{align*}
\begin{pmatrix}
1 & 2 & -1 & 3 \\
2 & 3 & -5 & 3 \\
5 & 8 & -11 & 9 \\
\end{pmatrix}
\end{align*}
\]

\[
-2R_1+R_2 = \begin{pmatrix}
1 & 2 & -1 & 3 \\
0 & -1 & -3 & -3 \\
5 & 8 & -11 & 9 \\
\end{pmatrix}
\]

\[
-5R_1+R_3 = \begin{pmatrix}
1 & 2 & -1 & 3 \\
0 & -1 & -3 & -3 \\
0 & -2 & -6 & -6 \\
\end{pmatrix}
\]

\[
-2R_2 = \begin{pmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & 3 & 3 \\
0 & -2 & -6 & -6 \\
\end{pmatrix}
\]

\[
2R_2+R_3 = \begin{pmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & 3 & 3 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Notice we are left with the system of equations below.

\[
\begin{align*}
x + 2y - z &= 3 \\
y + 3z &= 3 \\
\end{align*}
\]

This system of equations has 3 variables but only 2 equations. Therefore, there are infinitely many solutions. In this case, we will introduce the parameter \( z = a \).

\[
\begin{align*}
x + 2y - a &= 3 \\
y + 3a &= 3 \\
\end{align*}
\]

\[y = -3a + 3\]

\[
\begin{align*}
x + 2(-3a + 3) - a &= 3 \\
x - 6a + 6 - a &= 3 \\
x - 7a + 6 &= 3 \\
x &= 7a - 3 \\
\end{align*}
\]

Thus, for any real number \( a \), the point \((7a - 3, -3a + 3, a)\) is a solution to the system of equations.

Example 3.b. Solve the following systems of equations by performing Gaussian elimination and back-substitution.

\[
\begin{align*}
3x - y + 2z &= 4 \\
x - 5y + 4z &= 3 \\
6x - 2y + 4z &= -8 \\
\end{align*}
\]
\[
\begin{bmatrix}
3 & -1 & 2 \\
1 & -5 & 4 \\
6 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
4 \\
3 \\
\end{bmatrix}
\]

\[R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix}
1 & -5 & 4 \\
3 & -1 & 2 \\
6 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
3 \\
4 \\
\end{bmatrix}
\]

\[-3R_1 + R_2 \Rightarrow \begin{bmatrix}
1 & -5 & 4 \\
0 & 14 & -10 \\
6 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
3 \\
-5 \\
-8
\end{bmatrix}
\]

\[-2R_2 \Rightarrow \begin{bmatrix}
1 & -5 & 4 \\
0 & 1 & -\frac{5}{7} \\
0 & 28 & -20
\end{bmatrix}
\begin{bmatrix}
3 \\
-\frac{5}{14} \\
-26
\end{bmatrix}
\]

\[\frac{1}{14}R_2 \Rightarrow \begin{bmatrix}
1 & -5 & 4 \\
0 & 1 & -\frac{5}{7} \\
0 & 28 & -20
\end{bmatrix}
\begin{bmatrix}
3 \\
-\frac{5}{14} \\
-16
\end{bmatrix}
\]

\[-28R_2 + R_3 \Rightarrow \begin{bmatrix}
1 & -5 & 4 \\
0 & 1 & -\frac{5}{7} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
3 \\
-\frac{5}{14} \\
-16
\end{bmatrix}
\]

Notice we are left with the system of equations below.

\[
\begin{align*}
x - 5y + 4z &= 3 \\
y - \frac{5}{7}z &= -\frac{5}{14} \\
0 &= -16
\end{align*}
\]

This system of equations has no solution.
6 Conic Sections

6.1 Coordinate Geometry

In this section, we set out to prove an important formula – the *distance formula*. The distance formula is essential for discussing conics. To establish our distance formula, we must remember a very basic theorem from geometry.

**Proposition 1. The Pythagorean Theorem.** For any right triangle with legs of length \(a\) and \(b\) and hypotenuse of length \(c\),

\[
a^2 + b^2 = c^2.
\]

**Example 1.** A right triangle has legs of length 4 and 6. Find the length of the hypotenuse.

From the Pythagorean Theorem, observe:

\[
a^2 + b^2 = c^2
\]

\[
4^2 + 6^2 = c^2
\]

\[
16 + 36 = c^2
\]

\[
52 = c^2
\]

\[
\pm 2\sqrt{13} = c
\]

Since \(c\) represents the length of the hypotenuse, we may reject \(-2\sqrt{13}\) and conclude \(c = 2\sqrt{13}\).

**Theorem 1. The Distance Formula.** The distance between any two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

---

\(^3\)The reader may find a detailed discussion of the Pythagorean Theorem in numerous places. For a quick overview, I point the reader to the Wikipedia article: [http://en.wikipedia.org/wiki/Pythagorean_theorem](http://en.wikipedia.org/wiki/Pythagorean_theorem).
Proof: The proof follows directly from the Pythagorean Theorem.

\[ a^2 + b^2 = c^2 \]

\[(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2 \]

\[ \pm \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \]

Rejecting the negative value again, we arrive at \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \) as needed.

**Example 2.** Find the distance between the points \((-3, 4)\) and \((3, -1)\).

From the distance formula, observe:

\[ d = \sqrt{(3 - (-3))^2 + (-1 - 4)^2} \]

\[ = \sqrt{(6)^2 + (-5)^2} \]

\[ = \sqrt{36 + 25} \]

\[ = \sqrt{61} \]

**Proposition 2. The Midpoint Formula.** The point halfway between two points \((x_1, y_1)\) and \((x_2, y_2)\) is called the *midpoint* and is given by

\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right). \]

**Example 3.** Locate the midpoint between \((-3, 4)\) and \((3, -1)\).

The midpoint is given by

\[ \left( \frac{-3 + 3}{2}, \frac{4 + (-1)}{2} \right) \]

\[ = \left( 0, \frac{3}{2} \right) \]

**Example 4.** Verify the point given by the midpoint formula \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \) is equidistant from \((x_1, y_1)\) and \((x_2, y_2)\).
First, we will find the distance from \((\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})\) to \((x_1, y_1)\).

\[
d = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}
\]

\[
= \sqrt{\left(\frac{x_1 + x_2}{2} - 2x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - 2y_1\right)^2}
\]

\[
= \sqrt{\left(\frac{-x_1 + x_2}{2}\right)^2 + \left(\frac{-y_1 + y_2}{2}\right)^2}
\]

\[
= \sqrt{\frac{x_1^2 - 2x_1x_2 + x_2^2}{4} + \frac{y_1^2 - 2y_1y_2 + y_2^2}{4}}
\]

We observe that these distances are equal, as desired.

We may now calculate the distance from \((\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})\) to \((x_2, y_2)\).

\[
d = \sqrt{\left(\frac{x_1 + x_2}{2} - x_2\right)^2 + \left(\frac{y_1 + y_2}{2} - y_2\right)^2}
\]

\[
= \sqrt{\left(\frac{x_1 + x_2}{2} - 2x_2\right)^2 + \left(\frac{y_1 + y_2}{2} - 2y_2\right)^2}
\]

\[
= \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2}
\]

\[
= \sqrt{\frac{x_1^2 - 2x_1x_2 + x_2^2}{4} + \frac{y_1^2 - 2y_1y_2 + y_2^2}{4}}
\]

We observe that these distances are equal, as desired.
6.2 Circles

Definition 1. A circle is the set of all points that a fixed distance, called the radius, from a fixed point, called the center.

Proposition 1. The equation
\[ x^2 + y^2 = r^2 \]
defines a circle centered at the origin with a radius of \( r \).

Example 1. Graph \( x^2 + y^2 = 1 \).

Proposition 2. The equation
\[ (x - h)^2 + (y - k)^2 = r^2 \]
defines a circle centered at \((h, k)\) with a radius of \( r \).

Example 2. Graph \((x - 3)^2 + (y + 1)^2 = 9\).