### 6.3 Factoring Special Products and Factoring Strategies

#### Objectives
1. Factor perfect square trinomials.
2. Factor a difference of squares.
3. Factor a difference of cubes.
4. Factor a sum of cubes.
5. Use various strategies to factor polynomials.

In Section 5.3, we explored special products found by squaring binomials and by multiplying conjugates. In this section, we see how to factor those special products.

**Objective 1**  
**Factor perfect square trinomials.** A perfect square trinomial is the product resulting from squaring a binomial. Recall the following rules for squaring binomials that we developed in Section 5.3.

\[
(a + b)^2 = a^2 + 2ab + b^2 \\
(a - b)^2 = a^2 - 2ab + b^2
\]

To use these rules when factoring, we simply reverse them.

**Rules**  
**Factoring Perfect Square Trinomials**

\[
a^2 + 2ab + b^2 = (a + b)^2 \\
a^2 - 2ab + b^2 = (a - b)^2
\]

**Note:** In these perfect square trinomials, the first and last terms are squares and the middle term is twice the product of their square roots.
Your Turn 1  Factor.

a.  $4x^2 + 20x + 49$

b.  $9n^2 - 48n + 64$

c.  $16h^2 + 72hk + 81k^2$

d.  $2r^3u^2 - 20r^2u^2 + 50tu^2$
Objective 2: Factor a difference of squares. Another special product we considered in Section 5.3 was that of conjugates, which are binomials that differ only in the sign separating the terms, as in $3x + 2$ and $3x - 2$. Note that the product of conjugates is a difference of squares.

$$(3x + 2)(3x - 2) = 9x^2 - 4$$

This term is the square of $3x$. This term is the square of $2$.

The rule for multiplying conjugates that we developed in Section 5.3 is

$$(a + b)(a - b) = a^2 - b^2.$$

Reversing this rule tells us how to factor a difference of squares.

Rule: Factoring a Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$
Your Turn 2  Factor.

a. $x^2 - 36$  
b. $16h^4 - 49k^6$  
c. $24y^5 - 54y^3$  
d. $t^4 - 16$  
e. $(3m + n)^2 - 25$
Objective 3  Factor a difference of cubes. Another special form that we can factor is $a^3 - b^3$, which is a difference of cubes. A difference of cubes is the product of a binomial in the form $a - b$ and a trinomial in the form $a^2 + ab + b^2$.

$$
(a - b)(a^2 + ab + b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\
= a^3 - b^3
$$

Multiply.  

Simplify.

Our multiplication suggests the following rule for factoring a difference of cubes.

**Rule**  
**Factoring a Difference of Cubes**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
Your Turn 3

Factor.

a. \( 1 - t^3 \)

b. \( 54xy^3 - 128x \)

\[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]
**Objective 4** Factor a sum of cubes. A *sum of cubes* has the form $a^3 + b^3$ and can be factored using a pattern similar to that for the difference of cubes. A sum of cubes is the product of a binomial in the form $a + b$ and a trinomial in the form $a^2 - ab + b^2$.

\[
(a + b)(a^2 - ab + b^2) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \quad \text{Multiply.}
\]
\[
= a^3 + b^3 \quad \text{Simplify.}
\]

Our multiplication suggests the following rule for factoring a sum of cubes.

**Rule** Factoring a Sum of Cubes

\[
a^3 + b^3 = (a + b)(a^2 - ab + b^2)
\]
Your Turn 4  Factor.

a. $64 + y^3$   
b. $81n^5 + 24n^2$   
c. $(3m - 2n)^3 + 64$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Procedure  Factoring a Polynomial
To factor a polynomial, factor out any monomial GCF and then consider the number of terms in the polynomial.

I. Four terms: Try to factor by grouping.

II. Three terms: Determine whether the trinomial is a perfect square.
   A. If the trinomial is a perfect square, consider its form.
      1. If it is in the form \(a^2 + 2ab + b^2\), the factored form is \((a + b)^2\).
      2. If it is in the form \(a^2 - 2ab + b^2\), the factored form is \((a - b)^2\).
   B. If the trinomial is not a perfect square, consider its form.
      1. If it is in the form \(x^2 + bx + c\), find two factors of \(c\) whose sum is \(b\) and write the factored form as \((x + \text{first number})(x + \text{second number})\).
      2. If it is in the form \(ax^2 + bx + c\), where \(a \neq 1\), use trial and error. Or find two factors of \(ac\) whose sum is \(b\); write these factors as coefficients of two like terms that, when combined, equal \(bx\); and factor by grouping.

III. Two terms: Determine whether the binomial is a difference of squares, sum of cubes, or difference of cubes.
   A. If it is a difference of squares, \(a^2 - b^2\), the factors are conjugates and the factored form is \((a + b)(a - b)\). Note that a sum of squares cannot be factored.
   B. If it is a difference of cubes, \(a^3 - b^3\), the factored form is \((a - b)(a^2 + ab + b^2)\).
   C. If it is a sum of cubes, \(a^3 + b^3\), the factored form is \((a + b)(a^2 - ab + b^2)\).

*Note:* Always check to see if any of the factors can be factored further.
Your Turn 5  Factor completely.

a. $12m^4n + 27m^2n^3$  b. $6xz + 9xw - 8yz - 12yw$  c. $25a^2 + 30ab + 9b^2$
d. \[12a^2 - 7a - 10\]

\[12a^2 + 8a - 15a - 10\]

\[4a(3a+2) - 5(3a+2)\]

\[(3a+2)(4a-5)\]

\[\frac{a+b+c}{a(b+c)}\]

\[f. \ 16x^2 - 25\]

\[\text{Diff. of Squares}\]

\[a^2 - b^2 = (a+b)(a-b)\]

\[(4x+5)(4x-5)\]

\[\frac{8m^3 + 125n^3}{(2m)^3 + (5n)^3}\]

\[\frac{a^3 + b^3}{a^3 - ab + b^3}\]

\[\frac{a^2 - b^2}{a^2 - ab + b^2}\]

\[(2m)^2 = 4m^2\]

\[(5n)^2 = 25n^2\]

\[(2m+5n)(4m^2 - 10mn + 25n^2)\]
Your Turn 6  Factor completely.

a. $30tu^2 - 16tu + 2t$

b. $36x^2y - 12x^3y + x^4y$

c. $6m^3(n + 1)^2 - 24m^3$

d. $x^5 - 9x^3 + 27x^2 - 243$

e. $4x^2 - 12x + 9 - 36y^2$

\[
\begin{align*}
\text{Sum of Cubes} & \quad (a^3 + b^3) = (a + b)(a^2 - ab + b^2) \\
\text{Diff. of Squares} & \quad (a^2 - b^2) = (a + b)(a - b)
\end{align*}
\]
\[
4x^2 - 12x + 9 - 36y^2
\]

\[
(2x-3)^2 - (6y)^2
\]

Diff. of 2 Squares

\[
a^2 - b^2
\]

\[
2x-3 \quad 6y
\]

\[
(a-b)(a+b)
\]

\[
(2x-3 - 6y)(2x-3 + 6y)
\]