7.4 Solving Equations Containing Rational Expressions

Objective 1. Solve equations containing rational expressions.

To solve equations containing rational expressions, we multiply both sides of the equation by the LCD. Because all of the denominators divide evenly into the LCD, multiplying both sides of the equation by the LCD eliminates all of the denominators of all of the rational expressions in the equation. We then solve the resulting equation.

Procedure Solving Equations Containing Rational Expressions

To solve an equation containing rational expressions,

1. Eliminate the denominators of the rational expressions by multiplying both sides of the equation by the LCD of all the rational expressions in the equation.
2. Solve the resulting equation using the methods in Chapter 2 (for linear equations) and Chapter 6 (for quadratic equations).
3. Check your solution(s) in the original equation.

\[ \text{L.C.D.} = 15x \]
\[ \frac{4}{5} - \frac{1}{x} = \frac{2}{3} \]

\[ \text{LCM} \]
\[ \frac{4}{5} \]
\[ \frac{-1}{x} \]
\[ = \frac{2}{3} \]
\[ \left[ \frac{15x}{1} \right] \]

\[ 3(4x) - 15(1) = 2(5x) \]
\[ 12x - 15 = 10x \]
\[ 12x - 15 = 10x \]
\[ 2x = 15 \]
\[ x = \frac{15}{2} \]
Extraneous Solutions

If we multiply both sides of an equation by an expression that contains a variable, we might obtain a solution that, when substituted into the original equation, makes one of its denominators equal to 0 (and therefore makes one of its expressions undefined). We call such an apparent solution an *extraneous solution*, and we discard it.

By inspecting the denominator of each rational expression, you can determine the value(s) that would cause the expression to be undefined before you solve the equation.

\[
\text{Solve } \frac{5n}{n - 2} - 4 = \frac{10}{n - 2}
\]

\[
\begin{align*}
& n - 2 \neq 0 \\
& n \neq 2 \\
& \text{Not a possible solution}
\end{align*}
\]
Proportions

If an equation has the form \( \frac{a}{b} = \frac{c}{d} \), which is called a proportion, multiplying both sides by the LCD, \( bd \), gives the following:

\[
bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}
\]

\[
ad = bc
\]

A faster way to reach that same conclusion is to cross multiply.

**Warning:** Cross multiplication can be used only when the equation is of the form \( \frac{a}{b} = \frac{c}{d} \), a fraction equal to a fraction.

### Proportions and Their Cross Products

If \( \frac{a}{b} = \frac{c}{d} \), where \( b \neq 0 \) and \( d \neq 0 \), then \( ad = bc \).

Solve: \( \frac{y}{y-2} = \frac{10}{y+1} \)

**Cross Multiply**

\( y(y+1) = 10(y-2) \)

\( y^2 + y = 10y - 20 \)

\( y^2 - 9y + 20 = 0 \)

\( (y-4)(y-5) = 0 \)

\( y = 4 \) or \( y = 5 \)

\( LCD = (y-2)(y+1) \)

\( \frac{y}{y-2} = \frac{10}{y+1} \) \( (y-2)(y+1) \)

\( y(y+1) = 10(y-2) \)
Your Turn 4  Solve.

a. \( \frac{12}{x^2 - 4} = \frac{3}{x - 2} + \frac{1}{1} \) 

\[
\frac{(x+2)(x-2)}{1} \left[ \frac{12}{(x+2)(x-2)} \right] = \left[ \frac{3}{(x-2)} + \frac{1}{1} \right] \frac{(x+2)(x-2)}{1}
\]

\[
L.H.S. = \frac{12}{(x+2)(x-2)}
\]

Check

\[
12 = 3(x+2) + 1(x+2)(x-2)
\]

\[
\frac{12}{(x+2)(x-2)} = \frac{3(x+2) + 1(x+2)(x-2)}{x^2 - 2x + 2x - 4}
\]

\[
\frac{12}{(x+2)(x-2)} = \frac{3x + 6 + x^2 - 4}{x^2 - 4}
\]

\[
\frac{12}{(x+2)(x-2)} = \frac{x^2 + 3x + 2}{x^2 - 4}
\]

\[
12 = \frac{x^2 + 3x + 2}{x^2 - 4}
\]

\[
0 = x^2 + 3x - 60
\]

Factor

\[
0 = (x+5)(x-2)
\]

\[
\frac{12}{(x+2)(x-2)}
\]

\[
X = 2 \quad \text{Extraneous Root}
\]

\[
\frac{12}{0} = \frac{3}{0} + 1
\]

Undefined

\[
X = -5 \quad \text{or} \quad X = 2
\]

\[
X = -5
\]

\[
X = 2
\]

\[
X = -5
\]

\[
X = 2
\]
\[
\frac{7}{n^2 + 3n - 10} = \frac{6}{n^2 + 4n - 5} + \frac{3}{n^2 - 3n + 2}
\]

\[
\frac{(n+5)(n+2)(n-1)}{(n+5)(n+2)(n-1)} = \frac{6}{(n-1)(n+5)} + \frac{3}{(n-1)(n+2)}
\]

\[
7(n-1) = 6(n-2) + 3(n+5)
\]

\[
7n - 7 = 6n - 12 + 3n + 15
\]

\[
7n - 7 = 9n + 3
\]

\[
-2n = 2n + 3
\]

\[
-7 = 2n + 3
\]

\[
-10 = 2n
\]

\[
-5 = n
\]
Warning: Make sure you understand the difference between performing operations with rational expressions and solving equations containing rational expressions.

For example, an expression such as $\frac{2}{x} + \frac{3}{x + 1}$ can be evaluated or rewritten (not solved).

To rewrite this expression, we add the two rational expressions.

$$\frac{2}{x} + \frac{3}{x + 1} = \frac{2(x + 1)}{x(x + 1)} + \frac{3(x)}{(x + 1)(x)}$$

$$= \frac{2x + 2}{x(x + 1)} + \frac{3x}{x(x + 1)}$$

$$= \frac{5x + 2}{x(x + 1)}$$

Rewrite each rational expression with the LCD, $x(x + 1)$.

Multiply each numerator.

Add the numerators and keep the LCD.

An equation such as $\frac{2}{x} + \frac{3}{x + 1} = \frac{17}{12}$ can be solved. First, we eliminate the rational expressions by multiplying both sides of the equation by the LCD, $12x(x + 1)$.

$$12x(x + 1)\left(\frac{2}{x} + \frac{3}{x + 1}\right) = 12x(x + 1)\frac{17}{12}$$

Multiply both sides by the LCD.

$$12(x + 1)(2) + 12x(3) = x(x + 1)(17)$$

Simplify.

$$24x + 24 + 36x = 17x^2 + 17x$$

Multiply.

$$0 = 17x^2 - 43x - 24$$

Write in standard form.

$$0 = (17x + 8)(x - 3)$$

Factor.

$$x = -\frac{8}{17} \quad \text{or} \quad x = 3$$

Use the zero-factor theorem.