7.3 \#5

Simplify the complex rational fraction.

\[
\frac{\left(\frac{1}{1+\frac{1}{v}-\frac{6}{v^2}}\right)\left(\frac{v^2}{v}\right)}{\left(\frac{4}{v} + \frac{4}{v^2}\right)\left(\frac{v^2}{v}\right)} = \frac{v^2 + 4v - 6}{v^2 - 4v + 4} = \frac{(v^2 + 4v - 6)}{(v^2 - 4v + 4)} = \frac{(v+3)}{(v-2)}
\]

Method 2

\[\text{LCM} = v^2\]
For Exercises 7–18, simplify each rational expression.

18. \[
\frac{8x^3 + 27}{2x^2 - 7x - 15} = \frac{(2x^3 + 9)}{(x+3)} \cdot \frac{(4x^2 - 6x + 9)}{(2x - 3)} = \frac{(x-5)}{(x+3)}
\]

\[8x^3 + 27 = (2x+3)(4x^2 - 6x + 9)\]

\[a^3 + b^3 = (a+b)(a^2 - ab + b^2)\]

\[
\frac{2x^2 - 7x - 15}{2x^2 - 10x + 3x - 15} = \frac{2x(x-5) + 3(x-5)}{(x-5)(2x+3)}
\]

\[1.30\]

\[2.15\]

\[3.16\]

\[3.6\]
For Exercises 29–36, find each quotient.

33. \[\frac{4p^2 - 9}{24p + 28} \div \frac{10p - 15}{36p^2 - 49}\]

\[\frac{4p^2 - 9}{24p + 28} \times \frac{36p^2 - 49}{10p - 15}\]

\[\frac{(2p+3)(6p-7)}{4(6p+7)} \cdot \frac{(6p-7)}{5(2p-3)}\]

\[\frac{(2p+3)(6p-7)}{20}\]
37. Given \( f(x) = \frac{2x}{5-x} \), find

a. \( f(3) \)  

b. \( f(0) \)  

c. \( f(-1) \)

\[
f(x) = \frac{2x}{5-x}
\]

\[
f(3) = \frac{2(3)}{5-3} = \frac{6}{2} = 3
\]

\[
f(0) = \frac{2(0)}{5-0} = \frac{0}{5} = 0
\]

\[
f(-1) = \frac{2(-1)}{5-(-1)} = \frac{-2}{6} = -\frac{1}{3}
\]
[7.1] For Exercises 39 and 40, find the domain of the rational function.

40. \( f(x) = \frac{3x - 4}{x^2 + 4x - 12} \)

\[ x^2 + 4x - 12 \neq 0 \]
\[ (x+6)(x-2) \neq 0 \]
\[ x+6 \neq 0 \quad \text{or} \quad x-2 \neq 0 \]
\[ x \neq -6 \quad \text{or} \quad x \neq 2 \]
54. \( \frac{y - 3}{4y} - \frac{y + 2}{5y} \)

\[
\frac{(y-3)}{4y} \cdot \frac{5}{5} - \frac{(y+2)}{5y} \cdot \frac{4}{4}
\]

\[
\frac{5(y-3) - 4(y+2)}{20y}
\]

\[
\frac{5y - 15 - 4y - 8}{20y}
\]

\[
\text{LCM} = \frac{y - 23}{20y}
\]
Subtract in:

\[ \frac{4x - 3y}{3x - y} - \frac{3x - 4y}{y - 3x} \]

Flip All Signs To Top & Bottom

\[ \frac{3x - 4y}{y - 3x} \cdot \frac{(-1)}{(-1)} \]

\[ \frac{4x - 3y}{3x - y} - \frac{-3x + 4y}{3x - y} \]

\[ \frac{-3x + 4y}{3x - y} \]

\[ \frac{(4x - 3y) - (-3x + 4y)}{3x - y} = \frac{4x - 3y + 3x - 4y}{3x - y} \]

Simplify:

\[ \frac{7x - 7y}{3x - y} \]

\[ \frac{7(x - y)}{3x - y} \]
2.4 Solving Equation

\[
\frac{78}{v^2 - 5v + 6} - \frac{2}{v^2 - 2v - 8} = \frac{3}{v^2 - 16}
\]

\[
\Rightarrow \frac{v^2 + 5v + 6}{(v+2)(v+3)} = \frac{v^2 - 2v - 8}{(v-4)(v+2)}
\]

\[
\Rightarrow \frac{5(v+4)(v-4) - 2(v+3)(v+4)}{5(v^2 - 16)} = \frac{3(v^2 + 5v + 6)}{V^2 - 16}
\]

Set \(E_{uv} = 0\)

\[
5v^2 - 80 - 2v^2 - 12v - 24 = 3v^2 + 15v + 18
\]

\[
3v^2 - 17v - 104 = 3v^2 + 15v + 18
\]

\[
\Rightarrow (v+3)(v+4)(v-4)(v+2)
\]

\[
3v^2 - 15v + 104
\]

\[
-3v^2 - 15v + 104
\]

\[
-29v = 122
\]

\[
v = \frac{122}{-29}
\]
74. \[ \frac{12}{m-2} = 9 + \frac{m}{m-2} \]

\[ \frac{(m-2)}{1} \left[ \frac{12}{m-2} \right] = \left[ \frac{9}{1} + \frac{m}{m-2} \right] \frac{(m-2)}{1} \]

\[ 12 = 9(m-2) + m \]

\[ 12 = 9m - 18 + m \]

\[ 12 = 10m - 18 \quad +18 \]

\[ +18 \]

\[ \frac{30}{10} = \frac{10m}{10} \]

\[ 3 = m \]

\[ \text{Check} \]

\[ \frac{12}{3-2} = 9 + \frac{3}{3-2} \]

\[ \frac{12}{1} = 9 + 3 \]

\[ 12 = 12 \]
8.1 Radical Expressions and Functions

Objectives
1. Find the $n$th root of a number.
2. Approximate roots using a calculator.
3. Simplify radical expressions.
4. Evaluate radical functions.
5. Find the domain of radical functions.
6. Solve applications involving radical functions.

In this section through Section 8.5, we focus on the expression portion of our algebra pyramid and explore square root and radical expressions.

The Algebra Pyramid

- Radical Equations
  - $\sqrt{x} = 5$
  - $\sqrt{x+5} = \sqrt{x-7}$

- Radical Expressions
  - $\sqrt{50}$ or $\sqrt{x}$

- Constants and Variables
  - 2, 5, 7, x, y

Note: We explore radical equations in Section 8.6.
**Objective** 1 Find the *nth* root of a number. In Section 1.3, we learned that a square root of a given number is a number that, when squared, equals the given number. For example, a square root of 16 is 4 because $4^2 = 16$. However, another number can be squared to equal 16: $(-4)^2 = 16$; so $-4$ is also a square root of 16. Consequently, 16 has two square roots: 4 and $-4$. We can write them more compactly as $\pm 4$.

Similarly, a *cube root* of a given number is a number that when cubed, equals the given number. For example, the cube root of 8 is 2 because $2^3 = 8$.

The *fourth root* of a number is a number that when raised to the fourth power, equals the given number. For example, 3 is a fourth root of 81 because $3^4 = 81$. However, $(-3)^4 = 81$ also; so $\pm 3$ are fourth roots of 81. Our examples suggest the following definition of *nth* root.

**Definition** *nth* root: The number $b$ is an *nth* root of a number $a$ if $b^n = a$.

Recall from Section 1.3 that we used the symbol $\sqrt{}$, called a radical sign, to denote the square root of a number. To indicate roots other than square roots, we use the symbol $\sqrt[n]{a}$, read "the *nth* root of $a$," where $n$ is called the root index and indicates which root we are to find. If no root index is given, we assume that it is 2, the square root. The number $a$ is called the radicand and is the number or expression whose root we are to find. The entire expression is called a radical, and any expression containing a radical is called a radical expression.

We have seen some numbers that have more than one root, such as 16, which has two square roots, 4 and $-4$. To avoid confusion, the symbol $\sqrt[n]{a}$ means to find a single root. If $n$ is even, then $\sqrt[n]{a}$ denotes the nonnegative root only and is called the principal root. For example, the principal square root of 16 is 4. Further, for $\sqrt[n]{a}$ to exist as a real number when $n$ is even, $a$ must be nonnegative because no real number can be raised to an even power to equal a negative number. For example, $\sqrt{-4}$ does not exist as a real number because there is no real number whose square is $-4$.

If the root index is odd, the radicand can be positive or negative because a positive number raised to an odd power is positive and a negative number raised to an odd power is negative.

**Rule** Evaluating *nth* Roots

When evaluating a radical expression $\sqrt[n]{a}$, the sign of $a$ and the index $n$ will determine possible outcomes.

- **If** $a$ is nonnegative, then $\sqrt[n]{a} = b$, where $b \geq 0$ and $b^n = a$.
- If $a$ is negative and $n$ is even, then there is no real-number root.
- **If** $a$ is negative and $n$ is odd, then $\sqrt[n]{a} = b$, where $b$ is negative and $b^n = a$.
Your Turn 1  Evaluate each root, if possible.

a. \( \sqrt{121} \)  

b. \( -\sqrt{4} \)  

c. \( \sqrt{-64} \)  

d. \( \pm \sqrt{0.36} \)  

e. \( \sqrt{\frac{25}{36}} \)  

f. \( \sqrt[3]{64} \)  

g. \( \sqrt[3]{-27} \)  

h. \( -\sqrt{81} \)  

Principal Square Root

\[ \sqrt{121} = 11 \quad \Rightarrow 11^2 = 121 \]

\[ -\sqrt{4} = -2 \quad \Rightarrow -2 \cdot 2 = -4 \]

\[ \sqrt{-64} = \text{Not A Real } # \]

8 \cdot 8 = 64  

\( (-8)(-8) = 64 \)

\[ \pm \sqrt{0.36} = \pm 0.6 \quad \Rightarrow 0.6 \times 0.6 = 0.36 \]

\[ \sqrt{\frac{25}{36}} = \frac{\sqrt{25}}{\sqrt{36}} = \frac{5}{6} \quad \Rightarrow \frac{5}{6} \times \frac{5}{6} = \frac{25}{36} \]

\[ 3\sqrt{64} = 4 \quad \text{Because } 4 \times 4 \times 4 = 64 \]

\[ 3\sqrt{-27} = -3 \quad \text{Because } -3 \times (-3) \times (-3) = -27 \]

\[ -\sqrt[3]{81} = -3 \quad \text{Because } - (3 \times 3 \times 3) = -27 \]
Objective 2 Approximate roots using a calculator. All of the roots we have considered so far have been rational. Some roots, such as \( \sqrt{3} \), are called irrational. As we learned in Section 1.1, we cannot express the exact value of an irrational number using rational numbers. In fact, writing \( \sqrt{3} \) with the radical sign is the only way we can express the exact value. However, we may approximate \( \sqrt{3} \) using a calculator.

For example, a calculator shows that \( \sqrt{2} \approx 1.414213562 \), which we can round to various decimal places.

Approximating to two decimal places: \( \sqrt{2} \approx 1.41 \)
Approximating to three decimal places: \( \sqrt{2} \approx 1.414 \)

Note: Recall that \( \approx \) means "approximately equal to."

Example 2 Approximate the roots by using a calculator. Round to three decimal places.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( \sqrt{12} )</td>
<td>Answer ( \sqrt{12} \approx 3.464 )</td>
</tr>
<tr>
<td>b. ( -\sqrt{38} )</td>
<td>Answer ( -\sqrt{38} \approx -6.164 )</td>
</tr>
<tr>
<td>c. ( \sqrt[4]{45} )</td>
<td>Answer ( \sqrt[4]{45} \approx 3.557 )</td>
</tr>
</tbody>
</table>

Calculator Tips

To evaluate roots higher than a square root on a scientific calculator, use the \( \sqrt[n]{ \text{key}} \) key. First, type the index; then press \( \sqrt[n]{ \text{ and then the radicand. For example, to evaluate } \sqrt[4]{45}, \text{ type } 4 \sqrt[4]{ \text{ and then } 45}. \) On some graphing calculators, \( \sqrt[n]{ \) is a function in a menu accessed by pressing the \( \text{MATH} \) key.

Your Turn 2 Approximate the roots using a calculator. Round to three decimal places.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( \sqrt{19} )</td>
<td>b. ( -\sqrt{93} )</td>
</tr>
</tbody>
</table>
Objective 3 Simplify radical expressions. The definition of a root can also be used to find roots with variable radicands. Recall that with an even index, the principal root is nonnegative. So at first, we will assume that all variables represent nonnegative values. We will use \((a^m)^n = a^{mn}\) to verify the roots.

Note: Remember that with an even index, the principal root is nonnegative. Because we are assuming that the variables are nonnegative, our result accurately indicates the principal square root.

**Example 3** Find the root. Assume that variables represent nonnegative values.

<table>
<thead>
<tr>
<th>Example</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (\sqrt{x^2} = x)</td>
<td>Because ((x)^2 = x^2)</td>
</tr>
<tr>
<td>b. (\sqrt{a^3} = a^{3/2})</td>
<td>Because ((a^2)^2 = a^4)</td>
</tr>
<tr>
<td>c. (\sqrt{16x^8} = 4x^4)</td>
<td>Because ((4x^4)^2 = 16x^8)</td>
</tr>
<tr>
<td>d. (\sqrt[4]{25x^8} = \frac{5x^2}{7y})</td>
<td>Because ((\frac{5x^2}{7y})^2 = \frac{25x^8}{49y^2})</td>
</tr>
<tr>
<td>e. (\sqrt[6]{y^6} = y)</td>
<td>Because ((y^2)^3 = y^6)</td>
</tr>
<tr>
<td>f. (\sqrt[12]{16x^{12}} = 2x^3)</td>
<td>Because ((2x^3)^4 = 16x^{12})</td>
</tr>
</tbody>
</table>

Connection In parts c and f, notice the similarity between finding the root of a product and raising a product to a power. To raise a product to a power, we raise each factor to the power. To find a root of a product, we find the root of each factor.

\[\sqrt{x^{18}} = x^2 \quad \sqrt[5]{x^{36}} = x^6 \quad \sqrt[7]{x^{14}} = x^2\]
Your Turn 3  Find the root. Assume that variables represent nonnegative values.

a. \( \sqrt{x^4} \)  

b. \( \sqrt[4]{9x^{10}} \)  

c. \( \sqrt[3]{36a^{12}} \)  

d. \( \sqrt[4]{\frac{100x^4}{81y^6}} \)  

e. \( \sqrt[3]{27y^9} \)  

f. \( \sqrt[4]{b^8} \)  

\[
\sqrt{x^4} = x^2 \quad \sqrt[4]{9x^{10}} = 3x^5 \quad \sqrt[3]{36a^{12}} = 6a^6 \\
\sqrt[4]{\frac{100x^4}{81y^6}} = \frac{10x^2}{9y^3} \quad \sqrt[3]{27y^9} = 3y^3 \\
\sqrt[4]{b^8} = b^2 \quad \text{because} \quad b^2 \cdot b^2 \cdot b^2 \cdot b^2 = b^8 \\
\text{because} \quad (b^2)^4 = b^8
\]
If the variables can represent any real number and the index is even, we must be careful to ensure that the principal root is nonnegative by using absolute value symbols. If the root index is odd, however, we do not need absolute value symbols because the root can be positive or negative depending on the sign of the radicand.

\[
\sqrt{(-3)^2} = \sqrt{9} = 3
\]

\[
\sqrt{-(3)^2} = \sqrt{-9} = \text{Not a Real Number}
\]

**Note:** To illustrate why \( \sqrt{x^2} = |x| \), suppose we were to evaluate \( \sqrt{x^2} \) when \( x = -3 \). We would have \( \sqrt{(-3)^2} = \sqrt{9} = 3 \). Notice that the root, 3 is, in fact, the absolute value of \(-3\). If we had incorrectly stated that \( \sqrt{x^2} = x \), then \( \sqrt{(-3)^2} \) would have to equal \(-3\), which is not true.

**Your Turn 4** Find the roots. Assume that variables represent any real number.

a. \( \sqrt{x^4} \)  

\[
\sqrt{x^4} = |x^2| = x^2
\]

b. \( \sqrt{9x^{10}} \)  

\[
\sqrt{9x^{10}} = |3x^5| = 3|x|^5
\]

c. \( \sqrt[3]{36a^{12}} \)  

\[
\sqrt[3]{36a^{12}} = |6a^6| = 6a^6
\]

d. \( \sqrt[4]{1.21u^6t^2} \)  

\[
\sqrt[4]{1.21u^6t^2} = |1.1u^3t| = 1.1|u^3t|
\]

e. \( \sqrt[3]{\frac{100x^4}{81y^6}} \)  

\[
\sqrt[3]{\frac{100x^4}{81y^6}} = \left| \frac{10x^2}{9y^3} \right| = \frac{10x^2}{9}|y^3|
\]

f. \( \sqrt[3]{27y^9} \)  

\[
3\sqrt[3]{27y^9} = 3y^3 \text{ No}
\]

\[
3\sqrt[3]{27} = -3 \quad \text{Odd Roots Don't Need Absolute Value}
\]

\[
\sqrt[3]{(-3)^3} = \sqrt[3]{-27} = -3
\]
Objective Evaluate radical functions. Now that we have learned about radical expressions, let’s examine radical functions.

Definition Radical function: A function containing a radical expression whose radicand has a variable.

Example 5

a. Given \( f(x) = \sqrt{3x - 2} \), find \( f(3) \).

Solution To find \( f(3) \), substitute 3 for \( x \) and simplify.

\[
f(3) = \sqrt{3(3) - 2} = \sqrt{9 - 2} = \sqrt{7}
\]

Your Turn 5 Given \( f(x) = \sqrt{2x + 5} \), find each of the following

a. \( f(-1) \)  
b. \( f(-3) \)
Objective 6  Solve applications involving radical functions. Often radical functions appear in real-world situations where one variable is a function of another.

The velocity of a free-falling object is a function of the distance it has fallen. Ignoring air resistance, the velocity of an object, \( v \), in meters per second, can be found after it has fallen \( h \) meters by using the formula \( v = -\sqrt{19.6h} \). Find the velocity of a skydiver who jumps from a plane and puts her body into a dive position so that air resistance is minimized. Find her velocity after she falls 100 meters. Use the formula in Example 7(a).