6.1 Greatest Common Factor and Factoring by Grouping

Objectives
1. Find the greatest common factor of a set of terms.
2. Factor a monomial GCF out of the terms of a polynomial.
3. Factor polynomials by grouping.

Often in mathematics, we need a number or an expression to be in factored form.

Definition
Factored form: A number or an expression written as a product of factors.

For example, the following polynomials have been rewritten in factored form.

\[ 2x + 8 = 2(x + 4) \quad \text{Factored form} \]
\[ x^2 + 5x + 6 = (x + 2)(x + 3) \quad \text{Factored form} \]

Notice that we can check an expression's factored form by multiplying the factors to see if their product is the original expression. Writing factored form is called factoring.
Objective 1 Find the greatest common factor of a set of terms. When factoring polynomials, the first step is to determine whether there is a monomial factor that is common to all of the terms in the polynomial. Additionally, we want that monomial factor to be the greatest common factor of the terms.

Definition Greatest common factor (GCF) of a set of terms: A monomial with the greatest coefficient and degree that evenly divides all of the given terms.

For example, the greatest common factor of $12x^2$ and $18x^3$ is $6x^2$ because 6 is the greatest numerical value that evenly divides both 12 and 18 and $x^2$ is the highest power of $x$ that evenly divides both $x^2$ and $x^3$. Notice that $x^2$ has the smaller exponent, which provides a clue as to how to determine the GCF. We can use prime factorizations to help us find GCFs.

Procedure Finding the GCF
To find the GCF of two or more monomials,
1. Write the prime factorization in exponential form for each monomial. Treat variables like prime factors.
2. Write the GCF's factorization by including the prime factors (and variables) common to all of the factorizations, each raised to its smallest exponent in the factorizations.
3. Multiply the factors in the factorization created in step 2.

Note: If there are no common prime factors, the GCF is 1.
Your Turn 1

Find the GCF.

a. $32r^2t$ and $48r^3ts$

b. $35a^2$ and $9b$

Prime Numbers:

2, 3, 5, 7, 11, 13, 17, 19, 23

Factors:

32, 31, 37, 39, 41, 43

$GCF = 16r^2t$

$GCF = 2^4 = 16$

Common Factors:

Ave $2^4 = 16$
Objective 2  

Factor a monomial GCF out of the terms of a polynomial. Earlier we mentioned that $2(x + 4)$ is the factored form of $2x + 8$. Notice that 2 is the GCF of the terms $2x$ and 8. This suggests the following procedure for factoring a monomial GCF out of the terms of a polynomial.

Procedure

Factoring a Monomial GCF out of a Polynomial

To factor a monomial GCF out of the terms of a polynomial,

1. Find the GCF of the terms in the polynomial.
2. Rewrite the polynomial as a product of the GCF and the quotient of the polynomial and the GCF.

$$\text{Polynomial} = \text{GCF} \left( \frac{\text{Polynomial}}{\text{GCF}} \right)$$
Your Turn 2
Factor.

a. $12xy - 4x$
   $$4x(3y - 1)$$

b. $15m^3n - 21mn^2 + 27mnp$
   $$3mn(5m^2 - mn + 9p)$$
Factoring When the First Term Is Negative

Generally, we prefer the first term inside parentheses to be positive. To avoid a negative first term in the parentheses, factor out the negative of the GCF, if needed.

**Your Turn 3** Factor.

a. \(-40ab - 35b\)  
   \(-5b(8a + 7)\)

b. \(-30t^4u^5 - 24t^3u^4 + 48t^2uv\)
   \(-6t^2u^2(5t^2u^3 + 4tu^2 - 8v)\)
Factoring When the GCF Is a Polynomial

Sometimes when factoring, the GCF is a polynomial.

**Example 4**  Factor \( a(c + 5) + b(c + 5) \).

**Solution**  Notice that this expression is a sum of two products, \( a(c + 5) \) and \( b(c + 5) \). Further, note that \( c + 5 \) is the GCF of the two products.

\[
\begin{align*}
a(c + 5) + b(c + 5) &= (c + 5) \left( \frac{a(c + 5) + b(c + 5)}{c + 5} \right) \\
&= (c + 5)(a + b)
\end{align*}
\]

**Your Turn 4**  Factor \( 6n(m - 3) - 7(m - 3) \).

\((m - 3)(6n - 7)\)
Objective 3 Factor polynomials by grouping. Factoring out a polynomial GCF as we did in Example 4 is an intermediate step in a process called factoring by grouping, which is a technique that we try when factoring a four-term polynomial such as $ac + 5a + bc + 5b$. The method is called grouping because we group pairs of terms and look for a common factor each group. We begin by pairing the first two terms as one group and the last two terms as a second group.

$$ac + 5a + bc + 5b = (ac + 5a) + (bc + 5b)$$

Group pairs of terms.

Notice that the first two terms have a common factor of $a$ and the last two terms have a common factor of $b$. If we factor the $a$ out of the first two terms and the $b$ out of the last two terms, we have the same expression that we factored in Example 4.

$$= a(c + 5) + b(c + 5)$$

Factor out $a$ and $b$.

$$= (c + 5)(a + b)$$

Factor out $c + 5$.

Our example suggests the following procedure.

Procedure Factoring by Grouping

To factor a four-term polynomial by grouping,

1. Factor out any monomial GCF (other than 1) that is common to all four terms.
2. Group pairs of terms and factor the GCF out of each pair.
3. If there is a common binomial factor, factor it out.
4. If there is no common binomial factor, interchange the middle two terms and repeat the process. If there is still no common binomial factor, the polynomial cannot be factored by grouping.
Your Turn 5

Factor. In Pairs or By Grouping

a. $12x + 9bx + 8 + 6b$
   - Factors GCF from Pairs
   - $3x(4+3b) + 2(4+3b)$
   - $(4+3b)(3x+2)$

b. $3m^2 + 6m + 4mn + 8n$
   - $3m(m+2) + 4n(m+2)$
   - $(m+2)(3m+4n)$
c. \[40x^2y - 60x^2 - 8xy + 12x\]

\[4x(10xy - 15x - 2y + 3)\]

\[5x(2y - 3) - 1(2y - 3)\]

\[4x(2y - 3)(5x - 1)\]

Factor using \(A - B\)
factor out the GCF.

18. \(60ab + 80ac - 20a^2\)
factor by grouping. \[ 15c^3 - 5c^2d + 6cd^2 - 2d^3 \]
factor completely. \[10a^2b^2 - 10b^3 + 15a^2b - 15b^2\]