6.2 Factoring Trinomials

Objectives
1. Factor trinomials of the form $x^2 + bx + c$.
2. Factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, by trial.
3. Factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, by grouping.
4. Factor trinomials of the form $ax^2 + bx + c$ using substitution.

In this section, we learn to factor trinomials such as $x^2 + 5x + 6$ and $3x^2 - x - 10$.

Objective 1 Factor trinomials of the form $x^2 + bx + c$. First, we consider trinomials of the form $x^2 + bx + c$ in which the coefficient of the first (variable squared) term is 1. If a trinomial in this form is factorable, it will have two binomial factors. For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$. Let’s use FOIL to multiply $(x + 2)(x + 3)$ and look for patterns that will help us factor.

\[
(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6
\]

**Note:** The product of these two numbers is the last term, and their sum is the coefficient of the middle term.

This suggests the following procedure.
Procedure

Factoring $x^2 + bx + c$

To factor a trinomial of the form $x^2 + bx + c$,

1. Find two numbers with a product equal to $c$ and a sum equal to $b$.
2. The factored trinomial will have the form $(x + □)(x + □)$, where the second terms are the numbers found in Step 1.
Example 1

Factor.

\[ x^2 - 9x + 20 \]

\[ (x-4)(x-5) \]

What Are The Factors of The Constant 20 That Add To The Middle Coefficient

\[ b = 20 \]

Find the square root of 20:

\[ \sqrt{20} = 4.5 \]

Add 4 to the middle coefficient:

\[ a \cdot 4 = 20 \]

The factors of 20 are:

\[ (x-4)(x-5) \]
Note: If the sign of the last term of the trinomial is positive, both signs of the binomial factors will be the same as the sign of the middle term of the trinomial. If the sign of the last term of the trinomial is negative, the signs of the binomial factors will be opposites.

**Your Turn 1**

a. \( n^2 - 8n + 12 \)

\[ (n-2)(n-6) \]

-2 + -6 = -8
\((-2)(-6)=+12\)

b. \( y^2 + 10y - 24 \)

\[ (y-2)(y+12) \]

YES! -2 + 12 = 10
-2(-2) = -24

-2(-12) = 24

NO! +4 + 6 = 10
\( y(6) = 24 \)
Your Turn 2

Factor.

a. \(3n^3 - 18n^2 + 24n\)

\[3n(n^2 - 6n + 8)\]

\[3n(n-2)(n-4)\]

b. \(y^5 + 5y^4 - 18y^3\)

\[y^3(y^2 + 5y - 18)\]

No Factors of 18 That Add to 5

\[y^3(y^2 + 5y - 18)\]
Your Turn 4
Factor.

a. $x^2 + 9xy + 20y^2$

$\text{(x+4y)(x+5y)}$

b. $x^3 + 4x^2y - 21xy^2$
Objective 2: Factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, by trial. Now let’s factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, such as $5x^2 + 13x + 6$ and $6x^2 - x - 12$. To factor such trinomials, we must consider the factors of the $ax^2$ term and the factors of the $c$ term. First, we develop a trial-and-error method for finding the factored form.
Example 5  Factor $5x^2 + 13x + 6$.

**Solution**  Because all of the terms are positive, we know that all of the terms in the binomial factors will be positive.

The *first* terms must multiply to equal $5x^2$.
The *last* terms must multiply to equal 6.

These terms must be $5x$ and $x$.

$$5x^2 + 13x + 6 = (\phantom{x} + \phantom{x})(\phantom{x} + \phantom{x})$$

We now multiply various combinations of these *first* and *last* terms in binomial factors to see which pair gives the original trinomial. More specifically, we look at the sum of the *inner* and *outer* products to see which combination gives the correct middle term. Again, because 6 is positive and $13x$ is positive, we know that both last terms in the binomials must have plus signs.

\[
\begin{align*}
(5x + 1)(x + 6) & = 5x^2 + 30x + x + 6 = 5x^2 + 31x + 6 & \text{Incorrect combinations} \\
(5x + 6)(x + 1) & = 5x^2 + 5x + 6x + 6 = 5x^2 + 11x + 6 & \\
(5x + 2)(x + 3) & = 5x^2 + 15x + 2x + 6 = 5x^2 + 17x + 6 & \\
(5x + 3)(x + 2) & = 5x^2 + 10x + 3x + 6 = 5x^2 + 13x + 6 & \text{Correct combination}
\end{align*}
\]

**Answer**  $5x^2 + 13x + 6 = (5x + 3)(x + 2)$
Procedure  Factoring by Trial and Error

To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, by trial and error,

1. Look for a monomial GCF in all of the terms. If there is one, factor it out.

2. Write a pair of first terms whose product is $ax^2$.

\[
\begin{array}{c}
\text{ax}^2 \\
(\Box + \Box)(\Box + \Box)
\end{array}
\]

3. Write a pair of last terms whose product is $c$.

\[
\begin{array}{c}
\text{c} \\
(\Box + \Box)(\Box + \Box)
\end{array}
\]

4. Verify that the sum of the inner and outer products is $bx$ (the middle term of the trinomial).

\[
\begin{array}{c}
\text{Inner} \\
\text{Outer} \\
\text{bx}
\end{array}
\]

If the sum of the inner and outer products is not $bx$, try the following:

a. Exchange the last terms of the binomials from step 3; then repeat step 4.

b. For each additional pair of last terms, repeat steps 3 and 4.

c. For each additional pair of first terms, repeat steps 2–4.
Objective 3 Factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, by grouping. Because trial and error can be tedious, an alternative method is to factor by grouping, which we learned in Section 6.1. Recall that we grouped pairs of terms in a four-term polynomial, then factored out the GCF from each pair. Because a trinomial of the form $ax^2 + bx + c$ has only three terms, we split its $bx$ term into two like terms to create a four-term polynomial that we can factor by grouping.

To split the $bx$ term, we use the fact that if $ax^2 + bx + c$ is factorable, $b$ will equal the sum of a pair of factors of the product of $a$ and $c$. For example, in $5x^2 + 13x + 6$ from Example 5, the product of $a$ and $c$ is $(5)(6) = 30$; so we need to find two factors of 30 whose sum is $b$, 13. It is helpful to list the factor pairs and their sums in a table.

<table>
<thead>
<tr>
<th>Factors of $ac = 30$</th>
<th>Sum of Factors of $ac$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1)(30) = 30$</td>
<td>$1 + 30 = 31$</td>
</tr>
<tr>
<td>$(2)(15) = 30$</td>
<td>$2 + 15 = 17$</td>
</tr>
<tr>
<td>$(3)(10) = 30$</td>
<td>$3 + 10 = 13$</td>
</tr>
<tr>
<td>$(5)(6) = 30$</td>
<td>$5 + 6 = 11$</td>
</tr>
</tbody>
</table>

Notice that 3 and 10 form the only factor pair of 30 whose sum is 13.

Now we can write $13x$ as $3x + 10x$ or $10x + 3x$ and then factor by grouping.

$5x^2 + 13x + 6 = 5x^2 + 3x + 10x + 6$ or $5x^2 + 10x + 3x + 6$

$= x(5x + 3) + 2(5x + 3)$

$= (5x + 3)(x + 2)$

$= 5x(x + 2) + 3(x + 2)$

$= (x + 2)(5x + 3)$

Every trinomial factorable by grouping has only one factor pair of $ac$ whose sum is $b$, which suggests the following procedure.

Factoring $ax^2 + bx + c$, Where $a \neq 1$, by Grouping

To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, by grouping,

1. Look for a monomial GCF in all of the terms. If there is one, factor it out.
2. Find two factors of the product $ac$ whose sum is $b$.
3. Write a four-term polynomial in which $bx$ is written as the sum of two like terms whose coefficients are the two factors you found in step 2.
4. Factor by grouping.
Your Turn 7  Factor by grouping.

a. \(10x^2 - 19x + 6\)

1. Multiply 1st & 3rd Terms
   - \(60x^2\)

2. What are the factors of 60 that add to -19
   - \(-4\) and \(-15\)

3. Rewrite middle term using -4 and -15
   - \(2x(5x-2)-3(5x-2)\)

4. Factor by grouping
   - \((5x-2)(2x-3)\)
**Objective 4** Factor trinomials of the form $ax^2 + bx + c$ using substitution. Some rather complicated-looking polynomials are actually in the form $ax^2 + bx + c$ and can be factored using a method called substitution.

**Your Turn B** Factor using substitution.

a. $24(t+2)^2 - 22(t+2) + 3$

- **Product**
  - $72u^2$

- **Factors of 72**
  - That Add to $-22$
    - $72$
    - $1\cdot72$
    - $2\cdot36$
    - $3\cdot24$
    - $-4\cdot-18 = -22$
    - $(-4)(-18) = 72$

- **Substitute $t+2$ for $u$**
  - $24u^2 - 22u + 3$
    - Revisiting Factor Pair
  - $24u^2 - 4u - 18u + 3$
    - $4u(6u-1) - 3(6u-1)$
      - $(6u-1)(4u-3)$

- **Final Answer**
  - $[6(6+1)] [4(6+1)]$
    - $(6t+11)(4t+5)$
Factor using substitution.

\[ 15n^4 - 19n^2 + 6 \]
factor completely.

\[ n^2 - n - 30 \]
factor the trinomials containing two variables.

\[ a^2 - 10ab + 24b^2 \]
factor completely.

\[3m^2 - 10mn - 8n^2\]
factor using substitution.

$$12r^4 - 19r^2 + 5$$
b. $15n^4 - 19n^2 + 6$
Factor by grouping.

b. $36x^4y + 3x^3y - 60x^2y$
Now we can write $13x$ as $3x + 10x$ or $10x + 3x$ and then factor by grouping.

$$5x^2 + 13x + 6 = 5x^2 + 3x + 10x + 6 \quad \text{or} \quad 5x^2 + 10x + 3x + 6$$

$$= x(5x + 3) + 2(5x + 3) \quad = 5x(x + 2) + 3(x + 2)$$

$$= (5x + 3)(x + 2) \quad = (x + 2)(5x + 3)$$

Every trinomial factorable by grouping has only one factor pair of $ac$ whose sum is $b$, which suggests the following procedure.

**Factoring $ax^2 + bx + c$, Where $a \neq 1$, by Grouping**

To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, by grouping,

1. Look for a monomial GCF in all of the terms. If there is one, factor it out.
2. Find two factors of the product $ac$ whose sum is $b$.
3. Write a four-term polynomial in which $bx$ is written as the sum of two like terms whose coefficients are the two factors you found in step 2.
4. Factor by grouping.