\[ \sqrt{115} = 5 \sqrt{28} \]

Perfect Linear
\[ \sqrt{25 \sqrt{7}} = 5 \sqrt{4 \sqrt{7}} \]

\[ 5 \sqrt{7} - 5(2) \sqrt{7} \]

\[ 5 \sqrt{7} - 10 \sqrt{7} \]

\[ -5 \sqrt{7} \]
\[ \sqrt{1875} \]
\[ \sqrt{625} \sqrt{3} \]
\[ \sqrt{5^4} \sqrt{3} \]
\[ 5^2 \sqrt{3} \]
\[ \boxed{25 \sqrt{3}} \]

\[ \frac{1875}{5} \]
\[ 375 \]
\[ 75 \]
\[ 15 \]
\[ 3 \]
\[ 0 \]
\[ \boxed{5.5, 5.5, 3} \]
\[ 5.5 \]
\[
\sqrt{180} \\
\sqrt{223.3} \quad \sqrt{5} \\
\sqrt{56} \quad 15 \\
6\sqrt{5}
\]
8.5 Rationalizing Numerators and Denominators of Radical Expressions

Objectives

1. Rationalize denominators.
2. Rationalize denominators that have a sum or difference with a square root term.
3. Rationalize numerators.

We are now ready to formalize the conditions for a radical expression that is in simplest form. A radical expression is in simplest form if

1. The radicand has no factor raised to a power greater than or equal to the index.
2. There are neither radicals in the denominator of a fraction nor radicands that are fractions.
3. All possible sums, differences, products, and quotients have been found.

In this section, we explore how to simplify expressions that have a radical in the denominator of a fraction, as in \( \frac{1}{\sqrt{2}} \).
Objective 1 Rationalize denominators. If the denominator of a fraction contains a radical, our goal is to rationalize the denominator, which means to rewrite the expression so that it has a rational number in the denominator. In general, we multiply the fraction by a well-chosen 1 so that the radical is eliminated. We determine that 1 by finding a factor that multiplies the \( n \)th root in the denominator so that its radicand is a perfect \( n \)th power.

Square Root Denominators

In the case of a square root in the denominator, we multiply it by a factor that makes the radicand a perfect square, which allows us to eliminate the square root.

For example, to rationalize \( \frac{1}{\sqrt{2}} \), we could multiply by \( \frac{\sqrt{2}}{\sqrt{2}} \) because the product’s denominator is the square root of a perfect square.

\[
\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]

\[
\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}
\]

**Note:** We are not changing the value of \( \frac{1}{\sqrt{2}} \) because we are multiplying it by 1 in the form of \( \frac{\sqrt{2}}{\sqrt{2}} \).

Any factor that produces a perfect square radicand will work. For example, we could have multiplied \( \frac{1}{\sqrt{2}} \) by \( \frac{\sqrt{8}}{\sqrt{8}} \).

\[
\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{8}}{\sqrt{16}} = \frac{\sqrt{4 \cdot 2}}{4} = \frac{2 \sqrt{2}}{4} = \frac{\sqrt{2}}{2}
\]

Notice, however, that multiplying by \( \frac{\sqrt{8}}{\sqrt{2}} \) required fewer steps to simplify than multiplying by \( \frac{\sqrt{8}}{\sqrt{8}} \).
Your Turn 1  Rationalize the denominator. Assume that variables represent positive values.

a. \[
\frac{1}{\sqrt{7}} \cdot \frac{7}{11} = \frac{7}{11\sqrt{7}} = \frac{\sqrt{7}}{11}
\]

b. \[
\sqrt{\frac{7}{12}} = \frac{\sqrt{7}}{\sqrt{12}} \cdot \frac{\sqrt{12}}{\sqrt{12}} = \frac{\sqrt{84}}{12}
\]

\[
= \frac{\sqrt{84}}{\sqrt{4 \cdot 21}} = \frac{\sqrt{3 \cdot 28}}{\sqrt{4 \cdot 21}} = \frac{\sqrt{3} \cdot \sqrt{28}}{2 \cdot \sqrt{21}} = \frac{\sqrt{3} \cdot 2\sqrt{7}}{2 \cdot \sqrt{21}} = \frac{3\sqrt{7}}{2\sqrt{21}}
\]

\[
= \frac{3\sqrt{7}}{2\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{3\sqrt{7} \cdot \sqrt{21}}{2\sqrt{21} \cdot \sqrt{21}} = \frac{3\sqrt{3}}{2 \cdot 3} = \frac{\sqrt{3}}{2}
\]

Rationalizing A Denominator

C. \[
\frac{3}{\sqrt{10x}} \cdot \frac{\sqrt{10x}}{\sqrt{10x}} = \frac{3\sqrt{10x}}{10x}
\]

Due to}

\[
\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{3}} = \frac{2\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{15}}{3}
\]
**nth-Root Denominators**

If the denominator contains a higher-order root such as a cube root, we multiply appropriately to get a perfect cube radicand in the denominator so that we can eliminate the radical. For example, \( \frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{25}} \cdot \frac{\sqrt[3]{25}}{\sqrt[3]{125}} = \frac{2 \sqrt[3]{25}}{\sqrt[3]{125}} = \frac{2 \sqrt[3]{25}}{5} \). We summarize as follows.

**Procedure**  
**Rationalizing Denominators**

To rationalize a denominator containing a single \( n \)th root, multiply the fraction by a well-chosen 1 so that the product's denominator has a radicand that is a perfect \( n \)th power.

**Your Turn 2**  
Rationalize the denominator. Assume that variables represent positive values.

a. \( \frac{6}{\sqrt[3]{3}} \cdot \frac{\sqrt[3]{27}}{\sqrt[3]{9}} = \frac{6 \cdot \sqrt[3]{9}}{\sqrt[3]{3} \cdot \sqrt[3]{9}} = \frac{6 \cdot \sqrt[3]{9}}{\sqrt[3]{27}} = 3 \)

b. \( \frac{\sqrt[3]{3}}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{3x}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{3x}}{x} \)

c. \( \frac{4}{\sqrt[4]{4y}} \cdot \frac{\sqrt[4]{2y^2}}{\sqrt[4]{2y^2}} = \frac{4 \cdot \sqrt[4]{2y^2}}{\sqrt[4]{2y^2} \cdot \sqrt[4]{2y^2}} = \frac{4 \cdot \sqrt[4]{2y^2}}{2y} \)

d. \( \frac{7}{\sqrt[2]{2t}} \cdot \frac{\sqrt[2]{2t^2}}{\sqrt[2]{2t^2}} = \frac{7 \cdot \sqrt[2]{2t^2}}{\sqrt[2]{2t^2} \cdot \sqrt[2]{2t^2}} = \frac{7 \cdot \sqrt[2]{2t^2}}{2t} \)
**Objective 2** Rationalize denominators that have a sum or difference with a square root term.

In Example 4 of Section 8.4, we saw that the product of two conjugates containing square roots does not contain any radicals. Consequently, if the denominator of a fraction contains a sum or difference with a square root term, we can rationalize the denominator by multiplying the fraction by a 1 made up of the conjugate of the denominator. For example, to rationalize \( \frac{5}{7 - \sqrt{3}} \), we multiply by \( \frac{7 + \sqrt{3}}{7 + \sqrt{3}} \). Because \( 7 - \sqrt{3} \) and \( 7 + \sqrt{3} \) are conjugates, their product will not contain any radicals; so the denominator will be rationalized.

\[
\frac{5}{7 - \sqrt{3}} = \frac{5}{7 - \sqrt{3}} \cdot \frac{7 + \sqrt{3}}{7 + \sqrt{3}} = \frac{5(7 + \sqrt{3})}{(7)^2 - (\sqrt{3})^2} = \frac{35 + 5\sqrt{3}}{49 - 3} = \frac{35 + 5\sqrt{3}}{46}
\]

**Procedure** Rationalizing a Denominator Containing a Sum or Difference

To rationalize a denominator containing a sum or difference with at least one square root term, multiply the fraction by a 1 whose numerator and denominator are the conjugate of the denominator.

---

**Your Turn 3**

Rationalize the denominator and simplify. Assume that variables represent positive values.

\[
\begin{align*}
\frac{9}{\sqrt{5} + 2} & \quad \frac{(\sqrt{5}-2)}{\sqrt{5}-2} \\
\frac{9}{\sqrt{5} + 2} & \quad \frac{(\sqrt{5}-2)}{\sqrt{5}-2} \\
\frac{9 - 4}{\sqrt{25} - 2\sqrt{5} + 4 - 4} & \quad \frac{9\sqrt{5} - 18}{1} = \frac{9\sqrt{5} - 18}{1}
\end{align*}
\]

Rationalize The Denominator

---

---
b. \[ \frac{\sqrt{2}}{\sqrt{5} - \sqrt{3}} \] * One *

\[ \frac{\sqrt{2}}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} \]

\[ = \frac{\sqrt{2}}{25 - 3} \]

\[ = \frac{\sqrt{2}}{22} \]

c. \[ \frac{3}{\sqrt{x} + 4} \]