8.6 Radical Equations and Problem Solving

Objective 1 Use the power rule to solve radical equations.

We now explore how to solve radical equations.

Definition Radical equation: An equation containing at least one radical expression whose radicand has a variable.

The Algebra Pyramid

Note: We move upward on the Algebra Pyramid, making equations out of the expressions we’ve studied.

X + 3 = 7

Objective 1 Use the power rule to solve radical equations. To solve radical equations, we use a new principle of equality called the power rule.

Rule Power Rule for Solving Equations
If both sides of an equation are raised to the same integer power, the resulting equation contains all solutions of the original equation and perhaps some solutions that do not solve the original equation. That is, the solutions of the equation \( a^n = b^n \) are contained among the solutions of \( a = b \), where \( n \) is an integer.

Isolated Radicals
First, we consider equations such as \( \sqrt{x} = 9 \) in which the radical is isolated. When we use the power rule, we raise both sides of the equation to the same integer power as the root index, then use the principle \((\sqrt[n]{x})^n = x\), which eliminates the radical, leaving its radicand.

1. Isolate the radical
2. Square both sides

\[
\sqrt{x} = 9 \\
(\sqrt{x})^2 = 9^2 \\
x = 81 \\
\sqrt{81} = 9
\]
Example 1  Solve.

a. \( \sqrt{x} = 9 \)

SOLUTION \( (\sqrt{x})^2 = (9)^2 \)  
Because the root index is 2, we square both sides.

\[ x = 81 \]

Check \( \sqrt{81} \neq 9 \)

\[ 9 = 9 \quad \text{True} \]

b. \( \sqrt[3]{y} = -2 \)

SOLUTION \( (\sqrt[3]{y})^3 = (-2)^3 \)  
Because the root index is 3, we cube both sides.

\[ y = -8 \]

Check \( \sqrt[3]{-8} \neq 2 \)

\[ -2 = -2 \quad \text{True} \]

Extraneous Solutions

In Section 7.4, we learned that some equations have extraneous solutions, which are apparent solutions that do not actually make the original equation true. Using the power rule can sometimes lead to extraneous solutions, so it is important to check solutions. For example, watch what happens when we use the power rule to solve the equation \( \sqrt{x} = -9 \).

\[ (\sqrt{x})^2 = (-9)^2 \quad \text{Square both sides.} \]

\[ x = 81 \]

By checking 81 in the original equation, we see that it is extraneous.

\[ \sqrt{81} \neq -9 \]

\[ 9 = -9 \quad \text{This equation is false, so 81 is extraneous.} \]

In fact, \( \sqrt{x} = -9 \) has no real number solution because if \( x \) is a real number, then \( \sqrt{x} \) must be nonnegative.

\[ (\sqrt{x})^2 = (-9)^2 \]

\[ x = 81 \quad \text{No Solution} \]

Check \( \sqrt{81} = -9 \)
Your Turn 2  Solve.

a. \( \sqrt{x - 5} = 6 \)

\[
\left( \sqrt{x - 5} \right)^2 = (6)^2
\]

\[
x - 5 = 36
\]

\[
\frac{+5}{+5}
\]

\[
x = 41
\]

Check

\[
\sqrt{41 - 5} = 6
\]

\[
\sqrt{36} = 6
\]

\[
6 = 6
\]
b. $\sqrt[3]{x + 2} = 3$

\[
\begin{align*}
\sqrt[3]{x + 2}^3 &= (3)^3 \\
x + 2 &= 27 \\
-2 &\quad -2 \\
\hline
x &= 25
\end{align*}
\]

Check $\sqrt[3]{27} = 3$
c. \( \sqrt{x + 3} = -7 \)

\[
\left( \sqrt{x + 3} \right)^2 = (-7)^2
\]

\[
x + 3 = 49
\]

\[
x = 46
\]

Check:

\[
\sqrt{46 + 3} = -7
\]

\[
\sqrt{49} \neq -7
\]

No Solution
Radicals on Both Sides of the Equation

As we will see in Example 3, the power rule can be used to solve equations with radicals on both sides of the equal sign.

**Example 3**  Solve.

a. \( \sqrt{6x - 1} = \sqrt{x + 2} \)

**Solution**  
\[ (\sqrt{6x - 1})^2 = (\sqrt{x + 2})^2 \]

\[ 6x - 1 = x + 2 \]  
\[ 5x = 3 \]  
\[ x = \frac{3}{5} \]

Square both sides.  
Subtract \( x \) from and add 1 to both sides.  
Divide both sides by 5.

**Check**  
\[ \sqrt{6 \left( \frac{3}{5} \right) - 1} = \sqrt{\frac{3}{5} + 2} \]

\[ \sqrt{\frac{18}{5} - \frac{5}{5}} = \sqrt{\frac{3}{5} + \frac{10}{5}} \]

\[ \sqrt{\frac{13}{5}} = \sqrt{\frac{13}{5}} \]

True. The solution is \( \frac{3}{5} \).
Your Turn 3  Solve.

\( a. \ \sqrt{8x + 5} = \sqrt{2x + 7} \)

\[ (\sqrt{8x+5})^2 = (\sqrt{2x+7})^2 \]

\[ 8x + 5 = 2x + 7 \]

\[ -2x - 5 = -2x - 5 \]

\[ 6x = \frac{2}{6} \]

\[ x = \frac{1}{3} \]

\[ \sqrt{\frac{8}{3} + 5} = \sqrt{\frac{2}{3} \cdot \frac{1}{3} + 7} \]

\[ \sqrt{\frac{8}{3} + \frac{15}{3}} = \sqrt{\frac{2}{3} + \frac{21}{3}} \]

\[ \sqrt{\frac{23}{3}} = \sqrt{\frac{23}{3}} \]
b. \( (\sqrt[3]{6x + 9} \cdot \sqrt[3]{10x - 3})^3 \)  

Cube both sides

\[
6x + 9 = 10x - 3
\]

\[
-6x
\]

\[
9 = 4x - 3
\]

\[
+3 +3
\]

\[
12 = 4x
\]

\[
\frac{12}{4} = \frac{4x}{4}
\]

\[
3 = x
\]

Check

\[
\sqrt[3]{27} = 3\sqrt[3]{27}
\]
Multiple Solutions

Radical equations may have multiple solutions if, after using the power rule, we are left with a quadratic form.
Your Turn 4

Solve \( \sqrt{9x + 7} = (x + 3)^2 \)

\[ 9x + 7 = x^2 + 6x + 9 \]
\[ -9x - 7 = -x - 7 \]

Isolate Radical on One Side

FOIL

\((x+3)(x+3)\)

\[ x^2 + 3x + 3x + 9 \]

Factor

\[ 0 = x^2 - 3x + 2 \]
\[ 0 = (x-2)(x-1) \]

\[ x-2 = 0 \quad \text{or} \quad x-1 = 0 \]
\[ x = 2 \quad \text{or} \quad x = 1 \]

Check

\[ x = 2 \]
\( \sqrt{25} = 5 \)
\( 5 = 5 \)

\[ x = 1 \]
\( \sqrt{16} = 4 \)
\( 4 = 4 \)
Radicals Not Isolated

Now we consider radical equations in which the radical term is not isolated. In such equations, we must first isolate the radical term.
Your Turn 5  Solve.

a. \( \sqrt{5x^2 + 6x - 7} + 3x = 5x + 1 \)

\[
(\sqrt{5x^2 + 6x - 7})^2 = (2x + 1)^2
\]
b. $\sqrt[3]{3x + 6} - 7 = -4$  \textit{Isolate Radical}
Using the Power Rule Twice

Some equations may require that we use the power rule twice to eliminate all radicals.

**Your Turn 6** Solve $\sqrt{2x + 1} = \sqrt{x} + 1$
Procedure: Solving Radical Equations

To solve a radical equation,

1. Isolate the radical if necessary. (If there is more than one radical term, isolate one of the radical terms.)
2. Raise both sides of the equation to the same power as the root index of the isolated radical.
3. If all radicals have been eliminated, solve. If a radical term remains, isolate that radical term and raise both sides to the same power as its root index.
4. Check each solution. Any apparent solution that does not check is an extraneous solution.