5.4 Factoring Trinomials of the Form $ax^2 + bx + c, a \neq 1$

An Important Note

In this section, we discuss two methods of factoring trinomials of the form $ax^2 + bx + c, a \neq 1$. That is, we will be factoring trinomials whose squared term has a numerical coefficient not equal to 1, after removing any common factors. Examples of trinomials with $a \neq 1$:

- $2x^2 + 3x + 1 = (2x + 1)(x + 1)$
- $3x^2 + 4x + 1 = (3x + 1)(x + 1)$

The methods we discuss are (1) factoring by trial and error and (2) factoring by grouping. We present two different methods for factoring these trinomials because some students, and some instructors, prefer one method while others prefer the second method. You may use either method unless your instructor asks you to use a specific method. We will use the same examples to illustrate both methods so that you can make a comparison. Each method has advantages and disadvantages, but once you become familiar with the different methods, either factoring by trial and error or factoring by grouping you need only read the material related to that specific method. Factoring by trial and error was introduced in Section 5.3 and factoring by grouping was introduced in Section 5.2.

1. Factor Trinomials of the Form $ax^2 + bx + c, a \neq 1$, by Trial and Error

Let’s now discuss factoring trinomials of the form $ax^2 + bx + c, a \neq 1$, by the trial and error method, introduced in Section 5.3. It may be helpful for you to reread that material before going any further.

Recall that factoring is the reverse of multiplying. Consider the product of the following two binomials:

$$(2x + 3)(x + 5) = 2ax + 10x + 3x + 15 = 2ax + 13x + 15.$$ 

Notice that the product of the first terms of the binomials gives the $a$-squared term of the trinomial, $2x^2$. Also notice that the product of the last terms of the binomials gives the last term, or constant, of the trinomial, 15. Finally, notice that the sum of the products of the outer terms and inner terms of the binomials gives the middle term of the trinomial, 13x. When we factor a trinomial using trial and error, we make use of these important facts. Note that $2x^2 + 13x + 15$ in factored form is $(2x + 3)(x + 5)$.

$$2x^2 + 13x + 15 = (2x + 3)(x + 5).$$ 

When factoring a trinomial of the form $ax^2 + bx + c$ by trial and error, the product of the $x$-terms in the binomial factors must equal the first term of the trinomial, $ax^2$. Also, the product of the constants in the binomial factors, including their signs, must equal the constant, $c$, of the trinomial.

$$ax^2 + bx + c = (ax + b)(x + c).$$

For example, when factoring the trinomial $2x^2 + 7x + 6$, each of the following pairs of factors has a product of the first terms equal to $2x^2$ and a product of the last terms equal to 6:

<table>
<thead>
<tr>
<th>Trinomial</th>
<th>Possible Factors</th>
<th>Product of First Terms</th>
<th>Product of Last Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x^2 + 7x + 6$</td>
<td>$(2x + 1)(x + 6)$</td>
<td>$2x(x) = 2x^2$</td>
<td>$1(6) = 6$</td>
</tr>
<tr>
<td></td>
<td>$(2x + 2)(x + 3)$</td>
<td>$2x(x) = 2x^2$</td>
<td>$2(3) = 6$</td>
</tr>
<tr>
<td></td>
<td>$(2x + 3)(x + 2)$</td>
<td>$2x(x) = 2x^2$</td>
<td>$3(2) = 6$</td>
</tr>
<tr>
<td></td>
<td>$(2x + 6)(x + 1)$</td>
<td>$2x(x) = 2x^2$</td>
<td>$6(1) = 6$</td>
</tr>
</tbody>
</table>

Each of these pairs of factors is a possible answer, but only one has the correct factors. How do we determine which is the correct factoring of the trinomial $2x^2 + 7x + 6$? The key lies in the $x$-term. We know that when we multiply two binomials using the FOIL method the sum of the products of the outer and inner terms gives us the $x$-term of the trinomial. We use this concept in reverse to determine the correct pair of factors. We need to find the pair of factors whose sum of the products of the outer and inner terms is equal to the $x$-term of the trinomial.

$$ax^2 + bx + c = (ax + b)(x + c).$$

Now look at the possible pairs of factors we obtained for $2x^2 + 7x + 6$ to see if any yield the correct $x$-term, 7x.

<table>
<thead>
<tr>
<th>Trinomial</th>
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Since $(2x + 3)(x + 2)$ yields the correct $x$-term, 7x, the factors of the trinomial $2x^2 + 7x + 6$ are $(2x + 3)$ and $(x + 2)$.

$$2x^2 + 7x + 6 = (2x + 3)(x + 2).$$

We can check this factoring using the FOIL method.

$$2x^2 + 7x + 6 = 2x^2 + 3x + 2x^2 + 6x = 2x^2 + 6x + 3x + 6 = 2x^2 + 7x + 6.$$ 

Since we obtained the original trinomial, our factoring is correct.

Note: In the preceding illustration that $(2x + 1)(x + 6)$ are different factors than $(2x + 6)(x + 1)$, because in one case 1 is paired with 2x and in the second case 6 is paired with x. The factors $(2x + 1)(x + 6)$ and $(x + 6)(2x + 1)$ are, however, the same set of factors with their order reversed.
**Helpful Hint**

When factoring a trinomial of the form $ax^2 + bx + c$, remember that the sign of the constant, $c$, and the sign of the $x$-term, $bx$, offer valuable information. When factoring a trinomial by trial and error, first check the sign of the constant. If it is positive, the signs in both factors will be the same as the sign of the $x$-term. If the constant is negative, one factor will contain a plus sign and the other a negative sign.

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**To Factor Trinomials of the Form $ax^2 + bx + c$, $a \neq 1$, by Trial and Error**

1. Determine whether there is a factor common to all three terms. If so, factor it out.
2. Write all pairs of factors of the coefficient of the squared term, $a$.
3. Write all pairs of factors of the constant term, $c$.
4. Try various combinations of these factors until the correct middle term, $bx$, is found.

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6. $2x^2 + 9x + 4$

12. $3x^2 - 2x - 8$
2. **Factor Trinomials of the Form** $ax^2 + bx + c, a \neq 1$, **by Grouping**

We will now discuss the use of grouping. The steps in the box that follow give the procedure for factoring trinomials by grouping.

**To Factor Trinomials of the Form** $ax^2 + bx + c, a \neq 1$, **by Grouping**

1. Determine whether there is a factor common to all three terms. If so, factor it out.
2. Find two numbers whose product is equal to the product of $a$ times $c$, and whose sum is equal to $b$.
3. Rewrite the middle term, $bx$, as the sum or difference of two terms using the numbers found in step 2.
4. Factor by grouping as explained in Section 5.2.

8. $7x^2 + 37x + 10$

First: Take Product of 1st & Last Terms

$7 \times 10 = 70$

What Are The Factors of The Product, 70, That Add To The 1st, 37?

Factor by grouping.

$p^2 - 4p^2q + 6p^2q - 24pq^2$

1. 10
2. 35
3. 14

Rewrite Middle and Group

$2 + 35 = 37$

$2(35) = 70$

$(7x + 2)(x + 5)$

$(7x + 1)(x + 10)$

$(7x + n)(x + 1)$
26. \(7x^2 - 8x + 1\)

\[7x^2 - 7x - x + 1\]
\[7x(x-1) - 1(x-1)\]
\[= (x-1)(7x-1)\]
\[-18x^2\]

30. \(16z^2 - 8z + 1\)

\[-14z = -8\]
\[-1(6z) = +7\]

32. \(3z^2 - 11z - 6\)

Not Factorable

18

1. 18

-2 = 9

3. 6

-2 + 9 = -11

\((-2)(-9) = +18\)

Should Be -18

Not Going To Work!
22. $3a^2 + 7a - 20$

62. $24x^2 - 92x + 80$
64. \( 8m^2 + 4mn - 4n^2 \)

50. \( 300x^3 - 400x - 400 \)