THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by \( x = g(t) \) and \( y = h(t) \) such that C does not intersect itself on the interval \([a, b]\) (not necessarily at the endpoints), then the arc length of C over the interval is given by:

\[
\int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

In the preceding section you saw that if a circle rolls along a line, a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, the path of the point is an epicycloid. The next example shows how to find the arc length of an epicycloid.

EXAMPLE 4 Finding Arc Length

A circle of radius 1 rolls around the circumference of a larger circle of radius 4, as

\[
x = 4 \cos t - \cos 4t, \quad y = 4 \sin t - \sin 4t
\]

\[
S = \int_{0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_{0}^{2\pi} \sqrt{\left(-4 \sin t + 16 \sin 4t\right)^2 + \left(-4 \cos t - 16 \cos 4t\right)^2} \, dt
\]

\[
= \int_{0}^{2\pi} \sqrt{256 \sin^2 t + 256 \cos^2 t} \, dt
\]

\[
= \int_{0}^{2\pi} 16 \, dt
\]

\[
= 32\pi
\]

Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

THEOREM 10.9 Area of a Surface of Revolution

If a smooth curve C given by \( x = g(t) \) and \( y = h(t) \) do not cross itself on the interval \([a, b]\), then the area of the surface of revolution formed by revolving the curve around the coordinate axis is given by:

1. If the curve is revolved around the \( x \)-axis: \( A = \pi \int_{a}^{b} \left(y^2 \right) \, dt \)

2. If the curve is revolved around the \( y \)-axis: \( A = \pi \int_{a}^{b} \left(x^2 \right) \, dt \)

These formulas are easy to remember if you think of the differential of arc length as:

\[
dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Then the formulas are written as follows:

1. \( A = 2\pi \int_{a}^{b} y \, dx \)

2. \( A = 2\pi \int_{a}^{b} x \, dy \)