\[ f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{Geometric Series} \]

\[ q = 1 \quad r = x \]

\[ S = \frac{1}{1-x} \quad u = -x \quad du = -dx \]

\[ -\ln |1-x| + C \]

\[ C - S \sum_{n=0}^{\infty} x^n \]

\[ C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C - x - \frac{x^2}{2} - \frac{x^3}{3} + \ldots \]

Integrate this series.
\[ f(k) = \frac{3}{2^{-k}} \quad = \quad \frac{3}{4 + 2 - 4(x+\ell)} \]

\[ \sum_{n=0}^{\infty} \frac{1}{2} \left[ -\frac{3}{5} (x+1) \right]^n \]

\[ \frac{3}{6 - 4(x+1)} \quad \frac{3}{6} \quad \frac{9}{1 - \frac{4}{6} (x+1)} \quad \frac{9}{1 - r} \quad r = -\frac{2}{3} (x+1) \]

\[ -1 < \frac{2}{5} (x+1) < 1 \]
\[ -\frac{3}{2} < x+1 < \frac{2}{2} \]
\[ -\frac{3}{2} < x < \frac{1}{2} \quad (\text{if}) \]
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \]

\[ \sin(x^3) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} \]

\[ \int \sin(x^3) \, dx = \int \left( x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} \right) \, dx \]

\[ = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} \]

\[ .009 - .000005 \]
\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \\
\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} \\
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \text{General Term}
\]
Section 10.2

20. \[ x = y \sin 2\theta \]
   \[ y = 2 \cos 2\theta \]

\[ \sin^2(2\theta) + \cos^2(2\theta) \geq 1 \]

\[ \frac{x^2}{16} + \frac{y^2}{4} = 1 \]

\[ \left(\frac{x}{4}\right)^2 = (\sin(2\theta))^2 \]
\[ \left(\frac{y}{2}\right)^2 = (\cos(2\theta))^2 \]

\[ \frac{x^2}{16} \geq \sin^2(2\theta) \]
\[ \frac{y^2}{4} = \cos^2(2\theta) \]
Section 10.5

Trig Derivatives

\[ r = \left( 0.01 \right) \left( \frac{\theta}{20\pi} \right) \]

\[ 100\pi \leq \theta \leq 600\pi \]

\[ x = r \cos \theta \quad y = r \sin \theta \]

Parameter Equations

\[ \frac{dx}{d\theta} = -r \sin \theta \quad \frac{dy}{d\theta} = r \cos \theta \]

\[ \frac{dx}{d\theta} = \left[ \left( \frac{1}{200\pi} \right) \cos \theta + \left( \frac{\theta}{200\pi} \right) \left( -\sin \theta \right) \right] \]

\[ \frac{dy}{d\theta} = \left[ \left( \frac{1}{200\pi} \right) \sin \theta + \left( \frac{\theta}{200\pi} \right) \cos \theta \right] \]

\[ S = \int_{100\pi}^{600\pi} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \ d\theta \]

\[ \frac{1}{200\pi} \int_{100\pi}^{600\pi} \sqrt{1 + \left( \frac{\theta}{200\pi} \right)^2} \ d\theta \]

\[ \left( \sin \theta + \theta \cos \theta \right)^2 = \sin^2 \theta + 2\theta \sin \theta \cos \theta + \theta^2 \cos^2 \theta \]

\[ \left( \cos \theta - \theta \sin \theta \right)^2 = \cos^2 \theta - 2\theta \sin \theta \cos \theta + \theta^2 \sin^2 \theta \]

\[ \frac{1}{200\pi} \int_{100\pi}^{600\pi} \sqrt{1 + \theta^2} \ d\theta \]

\[ \frac{1}{200\pi} \left( \frac{1}{2} \right) \left[ \theta \sqrt{\theta^2 + 1} + \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right]_{100\pi}^{600\pi} \]

\[ \text{Result: } 2.74\pi \text{ in} \]
(2.3) \[ \begin{align*}
\chi &= t^2 - t \\
\gamma &= t^2 - 3t + 1
\end{align*} \]
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t-1}
\]

\[
t_2 = t^2 - t
\]

\[
t = \frac{3 - \sqrt{3}}{2}
\]

\[
t = \frac{3 + \sqrt{3}}{2}
\]

\[
t = -1
\]

\[
t = 2
\]

\[
\frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{9}{3} = 3
\]

\[
x = t^2 - t
\]

\[
y = t^2 - 3t - 1
\]
\( x = 2t + \text{cut} \quad y = 2t - \text{cut} \quad t > 0 \)

\[
\frac{dy}{dx} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{2 - \frac{t}{\text{cut}}}{2 + \frac{t}{\text{cut}}} = \frac{2\text{cut} - t}{2\text{cut} + t} = \frac{2\text{cut} - 1}{2\text{cut} + 1}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{\frac{2\text{cut} - 1}{2\text{cut} + 1}}{\frac{dx}{dt}} \right] = \frac{(2\text{cut} - 1)(2) - (2\text{cut} + 1)(2)}{(2\text{cut} + 1)^2} \cdot \frac{2\text{cut} + 1}{t} \cdot \frac{2\text{cut} + 1}{t}
\]

\[
= \frac{4\text{cut} + 2 - 4\text{cut} + 2}{(2\text{cut} + 1)^2} \cdot \frac{t}{(2\text{cut} + 1)}
\]

Always Positive \( \frac{4}{(2\text{cut} + 1)^2} \cdot \frac{t}{(2\text{cut} + 1)} \)

\[
\frac{d^2y}{dx^2} = \frac{4t}{(2\text{cut} + 1)^3} \quad t > 0
\]

So concave up.
Polar Coordinates

So far, you have been representing graphs as rectangular coordinate system. The corresponding equations can be written in either rectangular or parametric form. In this section, we consider a system called the **polar coordinate system**.

To form the polar coordinate system in the plane, pick a point called the origin, and construct from O an initial ray called the polar axis. Figure 10.36. Then each point P in the plane can be represented by an ordered pair (r, θ) as follows.

- \( r = \text{directed distance from } O \text{ to } P \)
- \( \theta = \text{directed angle, counterclockwise from pole to } P \)

Figure 10.37 shows three points on the polar coordinate system, it is convenient to locate points with respect to the origin by radial lines through the pole.
as follows.

\[ r = \text{directed distance from } O \text{ to } P \]
\[ \theta = \text{directed angle}, \text{ counterclockwise from polar axis to segment } \overline{OP} \]

Figure 10.37 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by radial lines through the pole.

![Diagram](image-url)

Figure 10.37

With rectangular coordinates, each point \((x, y)\) has a unique representation. This
With rectangular coordinates, each point \((x, y)\) has a unique representation. This is not true with polar coordinates. For instance, the coordinates \((r, \theta)\) and \((r, 2\pi + \theta)\) represent the same point [see parts (b) and (c) in Figure 10.37]. Also, because \(r\) is a directed distance, the coordinates \((r, \theta)\) and \((-r, \theta + \pi)\) represent the same point. In general, the point \((r, \theta)\) can be written as
\[
(r, \theta) = (r, \theta + 2n\pi)
\]
or
\[
(r, \theta) = (-r, \theta + (2n + 1)\pi)
\]
where \(n\) is any integer. Moreover, the pole is represented by \((0, \theta)\), where \(\theta\) is any angle.
CHAPTER 10 Conics, Parametric Equations, and Polar Coordinates

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, we start with the polar coordinate system where the origin is the pole, and the x-axis is the polar axis. (r, θ) represents a point in the polar coordinate system, and (x, y) represents the same point in the rectangular coordinate system.

The coordinates are related by:

\[
\begin{align*}
\tan \theta &= \frac{y}{x}, & \cos \theta &= \frac{x}{r}, & \sin \theta &= \frac{y}{r}.
\end{align*}
\]

If \( r < 0 \), you can show that the same relationships hold.

THEOREM 10.10 Coordinate Conversion

The polar coordinates \((r, \theta)\) of a point are related to the rectangular coordinates \((x, y)\) of the point as follows.
If $r < 0$, you can show that the same relationships hold.

**THEOREM 10.10 Coordinate Conversion**

The polar coordinates $(r, \theta)$ of a point are related to the rectangular coordinates $(x, y)$ of the point as follows.

1. $x = r \cos \theta$
2. $y = r \sin \theta$
3. $r^2 = x^2 + y^2$
4. $\tan \theta = \frac{y}{x}$

**EXAMPLE 1 Polar-to-Rectangular Conversion**

a. For the point $(r, \theta) = (2, \pi)$,

$$x = r \cos \theta = 2 \cos \pi = -2 \quad \text{and} \quad y = r \sin \theta = 2 \sin \pi = 0.$$ 

So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = (\sqrt{3}, \pi/6)$,

$$x = \sqrt{3} \cos \frac{\pi}{6} = \frac{3}{2} \quad \text{and} \quad y = \sqrt{3} \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$
\((-1, \frac{5\pi}{4})\)

\[r < 0\]

\[\theta = \frac{5\pi}{4}\]

\((-1, \frac{5\pi}{4})\)

\(\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)\)

\(\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)\)

Rectangular Coordinates

Polar

\(r < 0\)

\[x = r\cos \theta = (-1) \cos \left(\frac{5\pi}{4}\right) = (-1) \left(-\frac{\sqrt{2}}{2}\right)\]

\[y = r\sin \theta = (-1) \sin \left(\frac{5\pi}{4}\right) = (-1) \left(-\frac{\sqrt{2}}{2}\right)\]

\((-1, \frac{-\sqrt{2}}{2})\)

\((\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})\)
Try It! \((3i - 1)\) Rectangular

Find Polar Coordinates

\[ \tan \theta = \frac{y}{x} = \frac{-1}{3} \]

\[ r^2 = x^2 + y^2 \]

\[ r^2 = (3)^2 + (-1)^2 = 10 \]

\[ r = \pm \sqrt{10} \]

\[ \theta = \tan^{-1} \left( \frac{1}{3} \right) \]

\[ \theta = \tan^{-1} \left( \frac{-1}{3} \right) \]

\[ \theta = -32.17 \text{ radians} \]

\[ -32 + 2\pi = 5.26 \text{ radians} \]

\[ (\sqrt{10}, -32) = (\sqrt{10}, 5.26) \]

\[ (-\sqrt{10}, 2.8) \]

\[ -32 + \pi = 2.8 \]
EXAMPLE 2 Rectangular-to-Polar Conversion

a. For the second quadrant point \((x, y) = (-1, 1)\),

\[
\tan \theta = \frac{y}{x} = -1 \quad \rightarrow \quad \theta = \frac{3\pi}{4}.
\]

Because \(\theta\) was chosen to be in the same quadrant as \((x, y)\), you should use a positive value of \(r\).

\[
r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}.
\]

This implies that one set of polar coordinates is \((r, \theta) = (\sqrt{2}, 3\pi/4)\).

b. Because the point \((x, y) = (0, 2)\) lies on the positive y-axis, choose \(\theta = \pi/2\) and \(r = 2\), and one set of polar coordinates is \((r, \theta) = (2, \pi/2)\).

See Figure 10.40.
Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

**EXAMPLE 3** Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting the equation to rectangular form.

a. $r = 2$
   - **Solution**: Circle

b. $\theta = \frac{\pi}{3}$
   - **Line**

c. $r = \sec \theta$
   - **Vertical Line**

r cannot be 0, so the graph consists of all points that are a distance of 2 units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2. [See Figure 10.41(a).] You can confirm this by using the relations $r^2 = x^2 + y^2$ to obtain the rectangular equation

$$x^2 + y^2 = 2^2.$$  

**Rectangular equation**

b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis. [See Figure 10.41(b).] You
$r = 3 \cos \theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$3$</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Circle

\( r = 3 \cos \theta \)

\( r^2 = 3r \cos \theta \)

\( x^2 + y^2 = 3x \)

\( x^2 - 3x + y^2 = 0 \)

\( (x^2 - 3x + \frac{9}{4}) + (y^2) = \frac{9}{4} \)

\( (x - \frac{3}{2})^2 + (y - 0)^2 = \frac{9}{4} \)

O to 6

Give 5

One Revolution

Starts

At \((3, 0)\)
with the values of $\theta$ varying from $-4\pi$ to $4\pi$. This curve is of the form $r = a\theta$ and is called a \textbf{spiral of Archimedes}.

\section*{Conics, Parametric Equations, and Polar Coordinates}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example4}
\caption{Example 4: Sketching a Polar Graph}
\end{figure}

\textbf{EXAMPLE 4} Sketching a Polar Graph

Sketch the graph of $r = 2\cos 3\theta$.

\textbf{Solution} Begin by writing the polar equation in parametric form.

$$x = 2\cos 3\theta \cos \theta \quad \text{and} \quad y = 2\cos 3\theta \sin \theta$$

After some experimentation, you will find that the entire curve, which is called a \textbf{rose curve}, can be sketched by letting $\theta$ vary from 0 to $\pi$, as shown in Figure 10.43. If you try duplicating this graph with a graphing utility, you will find that by letting $\theta$ vary from 0 to $2\pi$, you will actually trace the entire curve \textit{twice}. 