Section 10.4

\[ \begin{align*}
& r \left( r = \sin \theta \right) r \\
& r^2 = r \sin \theta \\
& \downarrow \quad \downarrow \\
& \lambda^2 + y^2 = \gamma \\
& x^2 + y^2 - \gamma = 0 \\
& x^2(\gamma^2 - y + \frac{1}{4}) = \frac{1}{9} \\
& x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}
\end{align*} \]

Convert Back To Rectangular

\[ \begin{align*}
& r = \sin \theta \\
& 0 \leq \theta \leq \pi \\
& 1 \text{ Revolution}
\end{align*} \]
\[0 \leq t \leq \alpha\]

One Revolution
After some experimentation, you will find that the entire curve, which is called a rose curve, can be sketched by letting \( \theta \) vary from 0 to \( \pi \), as shown in Figure 10.43. If you try duplicating this graph with a graphing utility, you will find that by letting \( \theta \) vary from 0 to \( 2\pi \), you will actually trace the entire curve twice.
Slope and Tangent Lines

To find the slope of a tangent line to a polar graph, consider a differentiable function given by \( r = f(\theta) \). To find the slope in polar form, use the parametric equations

\[
\begin{align*}
    x &= r \cos \theta = f(\theta) \cos \theta \\
    y &= r \sin \theta = f(\theta) \sin \theta.
\end{align*}
\]

Using the parametric form of \( dy/dx \) given in Theorem 10.7, you have

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}
\]

which establishes the following theorem.

**THEOREM 10.11** Slope in Polar Form

If \( f \) is a differentiable function of \( \theta \), then the slope of the tangent line to the graph of \( r = f(\theta) \) at the point \((r, \theta)\) is

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}
\]

provided that \( dx/d\theta \neq 0 \) at \((r, \theta)\). (See Figure 10.45.)
THEOREM 10.11  Slope in Polar Form

If \( f \) is a differentiable function of \( \theta \), then the slope of the tangent line to the graph of \( r = f(\theta) \) at the point \((r, \theta)\) is

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}
\]

provided that \( dx/d\theta \neq 0 \) at \((r, \theta)\). (See Figure 10.45.)

From Theorem 10.11, you can make the following observations.

1. Solutions to \( \frac{dy}{d\theta} = 0 \) yield horizontal tangents, provided that \( \frac{dx}{d\theta} \neq 0 \).

2. Solutions to \( \frac{dx}{d\theta} = 0 \) yield vertical tangents, provided that \( \frac{dy}{d\theta} \neq 0 \).

If \( dy/d\theta \) and \( dx/d\theta \) are simultaneously 0, no conclusion can be drawn about tangent lines.

EXAMPLE 5  Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangent lines of \( r = \sin \theta \), \( 0 \leq \theta \leq \pi \).

Solution  Begin by writing the equation in parametric form.

\[
x = r \cos \theta = \sin \theta \cos \theta
\]

\[
y = r \sin \theta = \sin \theta \sin \theta
\]
\[ r = 1 + \sin \theta \]

Convert To Parametric Form

\[ x = r \cos \theta \quad y = r \sin \theta \]

\[ x = (1 + \sin \theta) \cos \theta \quad y = (1 + \sin \theta) \sin \theta \]

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta \cos \theta + (1 + \sin \theta) \cos \theta}{\cos \theta (\cos \theta + (1 + \sin \theta) (-\sin \theta)}
\]

\[
= \frac{\cos \theta (\sin \theta + (1 + \sin \theta))}{\cos^2 \theta - \sin \theta - \sin^2 \theta}
\]

\[
= \frac{\cos \theta (2 \sin \theta + 1)}{-2 \sin^2 \theta - \sin \theta + 1}
\]

\[ \frac{dx}{d\theta} = 0 \quad \text{Horizontal Tangents} \]

\[ \cos \theta (2 \sin \theta + 1) = 0 \]

\[ \cos \theta = 0 \quad \text{or} \quad 2 \sin \theta + 1 = 0 \]

\[ \cos \theta = 0 \quad \sin \theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \]

\[ \theta = \pm \frac{\pi}{3} \quad \text{or} \quad \pm \frac{5\pi}{6} \quad \text{Shapturn} \]

\[ \theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{Horizontal Tangents} \]

\[ \theta = -\frac{\pi}{6} \quad \text{or} \quad -\frac{5\pi}{6} \quad \text{Vertical Tangents} \]
\[ r = 1 + \sin t \]
\[ 0 \leq t \leq 2\pi \]

Our revolution
The distance to the function evaluated for the given angle.
Theorem 10.11 has an important consequence. Suppose the graph of \( r = f(\theta) \) passes through the pole when \( \theta = \alpha \) and \( f'(\alpha) \neq 0 \). Then the formula for \( \frac{dy}{dx} \) simplifies as follows.

\[
\frac{dy}{dx} = \frac{f'(\alpha) \sin \alpha + f(\alpha) \cos \alpha}{f'(\alpha) \cos \alpha - f(\alpha) \sin \alpha} = \frac{f'(\alpha) \sin \alpha + 0}{f'(\alpha) \cos \alpha - 0} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha
\]

So, the line \( \theta = \alpha \) is tangent to the graph at the pole, \((0, \alpha)\).

**THEOREM 10.12  Tangent Lines at the Pole**

If \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \), then the line \( \theta = \alpha \) is tangent at the pole to the graph of \( r = f(\theta) \).

Theorem 10.12 is useful because it states that the zeros of \( r = f(\theta) \) can be used to find the tangent lines at the pole. Note that because a polar curve can cross the pole more than once, it can have more than one tangent line at the pole. For example, the rose curve

\[ f(\theta) = 2 \cos 3\theta \]

has three tangent lines at the pole, as shown in Figure 10.48. For this curve, \( f(\theta) = 2 \cos 3\theta = 0 \) when \( \theta = \pi/6, \pi/2, \) and \( 5\pi/6 \). Moreover, the derivative \( f'(\theta) = -6 \sin 3\theta \) is not 0 for these values of \( \theta \).
Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the polar equation of a circle having radius $a$ and centered at the origin is simply $r = a$. Later in the text, you will see that this benefit. For now, several other types of graphs that have simple polar form are shown below. (Conics are considered in Section 10.4.)

**Limaçons**
\[
\begin{align*}
 r &= a \pm b \cos \theta \\
 r &= a \pm b \sin \theta \\
 (a > 0, b > 0)
\end{align*}
\]

**Rose Curves**
\[
\begin{align*}
 n \text{ petals if } n \text{ is odd} \\
 2n \text{ petals if } n \text{ is even} \\
 (n \geq 2)
\end{align*}
\]
Limaçons
\[ r = a \pm b \cos \theta \]
\[ r = a \pm b \sin \theta \]
\((a > 0, b > 0)\)

Rose Curves
\( n \) petals if \( n \) is odd
\( 2n \) petals if \( n \) is even
\((n \geq 2)\)

Circles and Lemniscates
The rose curves described above are of the form $r = a \cos n\theta$ or $r = a \sin n\theta$, where $n$ is a positive integer that is greater than or equal to 2. Use a graphing utility to graph $r = a \cos n\theta$ or $r = a \sin n\theta$ for some noninteger values of $n$. Are these graphs also rose curves? For example, try sketching the graph of $r = \cos 3\theta$, $0 \leq \theta \leq 6\pi$. 

TECHNOLOGY
Section 10.5

Area and Arc Length in Polar Coordinates

- Find the area of a region bounded by a polar curve.
- Find the points of intersection of two polar curves.
- Find the arc length of a polar graph.
- Find the area of a surface of revolution (polar coordinates).

Area of a Polar Region

The development of a formula for the area of a region on the rectangular coordinate system of rectangles as the basic element of area. If a circular sector of radius $r$ is given by $\frac{1}{2} \theta r^2$, then the area of a sector is $A = \theta r^2$. (Figure 10.49)
\[ A = \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \Delta \theta \]
\[ = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \]
which leads to the following theorem.

**THEOREM 10.13  Area in Polar Coordinates**

If \( f \) is continuous and nonnegative on the interval \([\alpha, \beta]\), then the area of the region bounded by the graph of \( r = f(\theta) \) for lines \( \theta = \alpha \) and \( \theta = \beta \) is given by

\[ A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \]
\[ = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta. \]

**NOTE** You can use the same formula to find the area of a region bounded by a continuous nonpositive function. However, the formula is not valid for the entire interval \( [\alpha, \beta] \) if the function is nonpositive.
Try It One

Area of One Petal

\[ r = 4 \sin 2 \theta \]

\[ \frac{1}{2} \int_{0}^{\frac{\pi}{3}} \left( 4 \sin 2\theta \right)^2 \, d\theta \]

\[ \frac{1}{2} \int_{0}^{\frac{\pi}{3}} 16 \sin^2 2\theta \, d\theta \]

\[ 8 \int_{0}^{\frac{\pi}{3}} \sin^2 2\theta \, d\theta \]

\[ 8 \int_{0}^{\frac{\pi}{3}} \left( \frac{1 - \cos 4\theta}{2} \right) \, d\theta \]

\[ 4 \int_{0}^{\frac{\pi}{2}} (1 - \cos 4\theta) \, d\theta \]

\[ u = 4\theta \quad du = 4\, d\theta \]

\[ 4 \left. \int_{0}^{\frac{\pi}{2}} (1 - \cos 4\theta) \, d\theta \right|_{0}^{\frac{\pi}{2}} \]

\[ 4 \theta - \sin 4\theta \]

\[ \left[ 4 \left( \frac{\pi}{2} \right) - \sin 2\pi \right] - \left[ 0 - 0 \right] \]

\[ 2\pi - 0 \]

\[ 2\pi \text{ Square units} \]
As $\theta$ goes from $0$ to $\frac{\pi}{2}$, one loop is traced. From $\frac{\pi}{2}$ to $\pi$, another loop is traced. From $\pi$ to $\frac{3\pi}{2}$, another loop is traced. From $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, another loop is traced. From $\frac{5\pi}{2}$ to $2\pi$, another loop is traced. Therefore, the function consists of 4 loops.
Try It 2

Area

Inside inner loop

\[ r = 3 + 4 \sin \theta \]

\[ \frac{1}{2} \int_{0}^{\frac{\pi}{2}} (3 + 4 \sin \theta)^2 \, d\theta \]

\[ \frac{1}{2} \int_{0}^{\frac{\pi}{2}} 9 + 24 \sin \theta + 16 \sin^2 \theta \, d\theta \]

\[ \frac{1}{2} \left[ 17 + 24 \sin \theta - 8 \cos \theta \right] \left[ \frac{\pi}{2} \right] \]

\[ \frac{1}{2} \left[ 17 \theta - 24 \cos \theta - 4 \sin 2\theta \right] \left[ \frac{\pi}{2} \right] \]

\[ \frac{1}{2} \left[ 17(5.43) - 24 \cos(5.43) - 4 \sin(2 \times 5.43) \right] \]

\[ - 17(3.99) - 24 \cos(3.99) - 4 \sin(2 \times 3.99) \]

\[ \frac{5.43 - 3.99}{5 \pi \text{ units}^2} \]

\[ \theta = -0.85 = 5.43 \]

\[ -0.85 + 2\pi = 5.43 \text{ radians} \]

\[ \pi + 0.85 = 3.99 \text{ radians} \]
Loop Is Plotted AS $\Theta$ Goes From 3.84 To 5.43 Radians

$\Theta = 3.84$  $\Theta = 5.73$
Try It 3

\[ r = 3 - 2\sin \theta \]

Double

\[ r = 3 - 2\cos \theta \]

\[
\frac{1}{2} \left[ \frac{1}{4} \int_{-\pi/2}^{\pi/2} (3 - 2\sin \theta)^2 \, d\theta \right]
\]

\[
\int (9 - 12\sin \theta + 4\sin^2 \theta) \, d\theta
\]

\[
9 \theta - 12\sin \theta \cos \theta + 4 \left( \frac{1 - \cos 2\theta}{2} \right) \, d\theta
\]

\[
9\theta + 12\cos \theta + 2\sin 2\theta - 2\frac{3}{4} \cos 2\theta \, d\theta
\]

\[
9\theta + 12\cos \theta + 2\theta - \sin 2\theta \left[ \frac{\sin \frac{\pi}{4}}{\frac{\pi}{4}} \right]
\]

\[
\left( 9\left( \frac{\pi}{4} \right) + 12\left( -\frac{\sqrt{2}}{2} \right) + 2\left( \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{2} \right) \right)
\]

\[
- \left( 9\left( \frac{\pi}{4} \right) + 12\left( -\frac{\sqrt{2}}{2} \right) + 2\left( \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{2} \right) \right) \approx 11\pi \approx 35.6
\]
**Arc Length in Polar Form**

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 77.)

**THEOREM 10.14  Arc Length of a Polar Curve**

Let $f$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

**EXAMPLE 4  Finding the Length of a Polar Curve**

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid
line $\theta = \pi/2$, as shown in Figure 10.57.

\[ r = \cos \theta \]

(a)

Figure 10.57

(b)

Solution  You can use the second formula given in Theorem 10.15 with $f'(\theta) = -\sin \theta$. Because the circle is traced once as $\theta$ increases from 0 to $\pi$, you have
Intersection

\[ r = 3 - 2\cos t \]

\[ 3 - 2\cos t = 3 - 2s \cos \theta \]

\[ -2\cos t = -2s \cos \theta \]

\[ \cos t = \sin \theta \]

\[ \theta = \frac{5\pi}{4}, \frac{3\pi}{4} \]

\[ r = 3 - 2\cos t \]

\[ \text{Double the shaded area} \]

\[ \theta = \frac{5\pi}{4}, r = 3 - 2s \cos \theta \]

\[ \frac{1}{2} \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} (3 - 2s \cos \theta)^2 d\theta \]