To sketch the graph of \( y = \ln x \), you can think of the rate of change at any point as the derivative of the natural logarithmic function given by the differential equation:

\[
\frac{dy}{dx} = \frac{1}{x}
\]

Figure 5.2 is a computer-generated graph, called a slope (or a direction) field, showing small line segments of slope \( \frac{1}{x} \). The graph of \( y = \ln x \) is the solution that passes through the point (1, 0). You will study slope fields in Section 5.1.

The following theorem lists some basic properties of the function.

**THEOREM 5.1 Properties of the Natural Logarithm Function**

The natural logarithmic function has the following properties:

1. The domain is \((0, \infty)\) and the range is \((-\infty, \infty)\).
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

**Proof.** The domain of \( f(x) = \ln x \) is \((0, \infty)\) by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

is positive for \( x > 0 \), as shown in Figure 5.3. It is concave up where its second derivative

\[
\frac{d^2}{dx^2} \ln x = -\frac{1}{x^2}
\]

is negative for \( x > 0 \). The proof that \( f \) is one-to-one is left as Exercise 111. The following limits imply that its range is the entire real line.

\[
\lim_{x \to \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \to 0^+} \ln x = -\infty
\]

Verification of these two limits is given in Appendix A.

Using the definition of the natural logarithmic function, important properties involving operations with natural logs are familiar with logarithms, you will recognize that these properties of all logarithms.
Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

**Theorem 5.2: Logarithmic Properties**

If $a$ and $b$ are positive numbers and $x$ is rational, then the following properties are true:

1. $\ln(1) = 0$
2. $\ln(ab) = \ln(a) + \ln(b)$
3. $\ln(e^x) = x \ln(e)$
4. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

**Example:**

Let $x = \frac{1}{2}$, $y = \frac{1}{3}$.

Then $\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{3}\right)$.

---

**Section 5.1: The Natural Logarithmic Function**

The Number $e$

It is likely that you have studied logarithms in an algebra course, but the logarithms would have been defined in $a$. For example, common logarithms have a base of 10 and $\log_{10}$.

For the natural logarithms, the base of $e$ is defined as the inverse of the logarithmic function. The natural logarithmic function is continuous, one-to-one, and has a unique real number $x$ such as $\ln(x) = 1$, and this number is denoted by the letter $e$. It can be shown that $e$ is the following decimal approximation:

$$e \approx 2.71828182846$$

**Definition of $e$**

The letter $e$ denotes the positive real number such that

$$\ln(e) = \frac{1}{2}$$

The value of $e$ is approximately $2.718$. This number is used in various mathematical formulas and applications, particularly in calculus and exponential growth or decay problems.
will learn more about this in Section 5.5.

The base for the natural logarithm is defined using the fact that the natural logarithmic function is continuous, one-to-one, and has a range of $(-\infty, \infty)$.

There must be a unique real number $e$ such that $\ln e = 1$, as shown in Figure 5.5. The number $e$ is denoted by the letter $e$.

It can be shown that $e$ is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

Definition of $e$

The letter $e$ denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t} \, dt = 1.$$

$$e^0 = 1$$

$$e^1 = e$$

FOR FURTHER INFORMATION: To learn more about the "Unsought Occurrences of the Number $e$" by Harris S. Shultz and Bill Leonard in Math Horizons Magazine.

Once you know that $\ln e = 1$, you can use logarithms to solve natural logarithms of several other numbers. For example, $\ln e^x = \ln (e^x) = x \cdot \ln e = x$.

You can evaluate $\ln(e^x)$ for various values of $x$, as shown in the table above.

The logarithms shown in the table above are convenient integer powers of $e$. Most logarithmic expressions are, however, decimals.
The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

**Theorem 5.3** Derivative of the Natural Logarithmic Function

Let $u$ be a differentiable function of $x$.

1. $\frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx}$, $u > 0$
2. $\frac{d}{dx}(\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx}$

**Example 2** Differentiation of Logarithmic Functions

Given $u = x^3 + 3$ where $x > 0$.

- $\frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx}$
- $\frac{d}{dx}(\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx}$
- $\frac{du}{dx} = 2x$

Hence, $f'(x) = \frac{1}{x^3 + 3} \cdot 2x = \frac{2x}{x^3 + 3}$

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NOTE: In Examples 4 and 5, we saw how the benefit of applying logarithmic properties before differentiating. Consider, for instance, the difficulty of direct differentiating the function given in Example 5.

On occasion, it is convenient to use logarithms on both sides in differentiating nonpolynomial functions. This procedure is called logarithmic differentiation.

EXAMPLE 4 Logarithmic Differentiation

Find the derivative of

\[ y = \frac{x - 2^x}{\sqrt{x^2 + 1}} \quad x \neq 2. \]

Solution: Note that \( y = 0 \) for all \( x \neq 2 \). So, \( y \) is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for \( y' \).

\[
\ln y = \ln \left( \frac{x - 2^x}{\sqrt{x^2 + 1}} \right)
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{\frac{d}{dx} \left( x - 2^x \right) - \frac{d}{dx} \left( \sqrt{x^2 + 1} \right)}{\sqrt{x^2 + 1}}
\]

\[
\frac{dy}{dx} = y \left( \frac{\frac{d}{dx} \left( x - 2^x \right) - \frac{d}{dx} \left( \sqrt{x^2 + 1} \right)}{\sqrt{x^2 + 1}} \right)
\]

\[
\frac{dy}{dx} = \frac{x - 2^x - \frac{1}{2} (x^2 + 1)^{-1/2} (2x)}{\sqrt{x^2 + 1}}
\]

\[
\frac{dy}{dx} = \frac{x(2^x - 1) - 1}{\sqrt{x^2 + 1}}
\]