Section 6.1

Prelim. Proj.

\[ y = x^2 + c e^{x^2} \]

\[ y' = 2x(c e^{x^2}) + e^{x^2} \]

\[ y' = x^2 + 2x e^{x^2} + e^{x^2} \]

\[ xy' - 2y = x^2 e^x \]

\[ x y' + 2x^2 e^x - 2 y = x^2 e^x \]

\[ x^2 e^x = x^2 e^x \]

Section 6.2

Mendlin
Section 6.3

Separation of Variables

Consider a differential equation that can be written in the form

$M(x) + N(y)\frac{dy}{dx} = 0$

where $M$ is a continuous function of $x$ alone and $N$ is a continuous function of $y$ alone.

As you saw in the preceding section, for this type of equation, all $x$ terms are collected with $dx$ and all $y$ terms with $dy$, and a solution can be obtained by integration. Such equations are said to be separable, and the solution procedure is called separation of variables. Below are some examples of differential equations that are separable.

<table>
<thead>
<tr>
<th>Original Differential Equation</th>
<th>Rewritten with Variables Separated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 3y \frac{dy}{dx} = 0$</td>
<td>$3y \frac{dy}{dx} = -x^2 \frac{dx}{x}$</td>
</tr>
<tr>
<td>$(\sin x)^2 = \cos x$</td>
<td>$\frac{1}{\cos x} \frac{dy}{dx} = \frac{1}{\sin x} \frac{dx}{x}$</td>
</tr>
<tr>
<td>$x^2 y' = 2$</td>
<td>$\frac{dy}{dx} = \frac{2}{x}$</td>
</tr>
</tbody>
</table>

Find General Solution of

$y' + 2e^x = 0$

$y' = -2e^x$

Integrate

$y = -2e^x + C$

$y = te^x + C$

General Solution
A common problem in electrostatics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6 shows a family of circles

\[ x^2 + y^2 = C \]

each of which intersects the lines in the family

\[ y = k \]

at right angles. Two such families of curves are said to be mutually orthogonal, and each curve in one of the families is called an orthogonal trajectory of the other family. In electrostatics, lines of force are orthogonal to the equipotential curves. In thermodynamics, lines of heat across isothermal curves. In hydrodynamics, the trajectories of the velocity potential curves.

**EXAMPLE 8 Finding Orthogonal Trajectories**

Describe the orthogonal trajectories for the family of curves given by

\[ y = k \]
Logistic Differential Equations

In Section 6.2, the exponential growth of a variable \( y \) is studied using a differential equation \( \frac{dy}{dt} = ky \). Growth is unlimited, but when the population size exceeds a certain limit, growth slows down or stops. A model that is of differential equation

\[
\frac{dy}{dt} = ky \left(1 - \frac{y}{L}\right)
\]

where \( k \) and \( L \) are positive constants, \( k \) represents the growth rate, and \( L \) is the maximum population size. The population increases when \( \frac{dy}{dt} > 0 \) and decreases when \( \frac{dy}{dt} < 0 \).

From Example 9, you can conclude that all solutions of the logistic equation are of the general form

\[
y = \frac{L}{1 + Ae^{-kt}}.
\]
EXAMPLE 10  Solving a Logistic Differential Equation

A state game commission released 20 elk into a game refuge. After 5 years, the elk population is 120. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population $p$ is

$$ \frac{dp}{dt} = p\left(1 - \frac{p}{4000}\right), \quad 0 \leq p \leq 4000 $$

where $t$ is the number of years.

a. Write a model for the elk population in terms of $t$.

b. Graph the slope field of the differential equation and the solution that passes through the point $(0, 20)$. 

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \rightarrow \infty$.

Solution

a. You know that $I_0 = 4000$. So, the solution of the equation is of the form