 Sequences
In mathematics, the word “sequence” is used in much the same way as in ordinary English. To say that a collection of objects is arranged in sequence usually means that the order is important. For example, in following a recipe, the order in which ingredients are added is very important. In mathematics, a sequence is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequence by some notation rather than by the standard function notation. For instance, in the sequence,

1, 2, 3, 4, 5, ...

Example 1: Finding the Terms of a Sequence
The terms of the sequence \( a_n = (1 + 1 - 1)^n \) are
1, 2, 3, 4, ...

Try It One Section 9.1

\[ a_1 = \frac{1}{2} \]
\[ a_2 = \frac{2}{3} \]
\[ a_3 = \frac{3}{4} \]
\[ a_4 = \frac{4}{5} \]
\[ a_n = \frac{n}{n+1} \]

General Term

Limit of The Sequence is One

b. The terms of the sequence \( b_n = \frac{1}{n} \)

1, 2, 3, 4, ...

d. The terms of the recursively defined sequence \( c_n \), where \( c_1 = 2 \) and \( c_{n+1} = c_n - 3 \) are

20, 25, 30, 35, 40, ...

Try It Exploration 1
EXAMPLE 5: Finding the Limit of a Sequence

Let \( \{a_n\} \) be a sequence such that \( f(n) = a_n \) for every positive integer \( n \), then

\[
\lim_{n \to \infty} a_n = L.
\]

NOTE: There are different ways to check a sequence one of which is to use the limit. One way is that the terms of the sequence become without bound or continue to increase without bound or decrease without bound.

Some increase without bound:

Term increase without bound:

\[
a_n = \frac{1}{n}.
\]

Some decrease without bound:

Term decrease without bound:

\[
a_n = \frac{1}{n^2}.
\]

Solution: In Theorem 5.5, you learned that

\[
\lim_{n \to \infty} \left( \frac{1}{n^2} \right) = 0.
\]

So, you can apply Theorem 5.5 to conclude that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0.
\]

Try It! 2

\[
q_n^3 = \frac{3n^3}{2n^3}.
\]

\( q_1 = \frac{3}{2} = 1 \)

\( q_2 = \frac{6}{5} \)

\( q_3 = \frac{3}{2} \)

\( q_4 = \frac{12}{7} = \frac{3}{2} \)

Limit on value of the terms of the sequence

Try It! 3

\[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = \frac{\infty}{\infty} \text{ Use L'Hopital's Rule}
\]

\( q_1 = \frac{1}{2} \)

\( q_2 = \frac{1}{3} \)

\( q_3 = \frac{1}{5} \)

\( q_4 = \frac{1}{7} \)

The terms of the sequence approach one.
The completeness axiom is used in the proof of Theorem 1.3.

**THEOREM 1.3: Bounded Monotone Sequences**

If a sequence \( \{a_n\} \) is bounded and monotone, then it converges.

**Proof:** Assume that the sequence is non-decreasing, as shown in Figure 1. For the sake of simplicity, also assume that each term in the sequence is unique. Because the sequence is bounded, there must exist an upper bound \( M \) such that

\[ a_n \leq M \text{ for all } n. \]

From the completeness axiom, it follows that there is a least upper bound \( L \) such that

\[ a_n < L \text{ for all } n \text{ and } a_n \rightarrow L. \]

For \( \epsilon > 0 \), it follows that \( L - \epsilon < L \), and therefore \( L - \epsilon \) cannot be an upper bound for the sequence. Consequently, at least one term \( a_m \) is greater than \( L - \epsilon \). Therefore, \( L - \epsilon < a_m \), and by the completeness axiom, there exists a term \( a_n \) such that \( a_n > L - \epsilon \). If \( a_n > M \), then this term is a contradiction, if not, then \( a_n \rightarrow L \). You may now know the least upper bound \( L \), and hence, \( L = \lim a_n \).

Example: A monotone sequence

- \( a_1 = 1 \)
- \( a_2 = 2 \)
- \( a_3 = 3 \)
- \( a_4 = 4 \)
- \( a_5 = 5 \)
- \( a_6 = 6 \)
- \( a_7 = 7 \)
- \( a_8 = 8 \)
- \( a_9 = 9 \)
- \( a_{10} = 10 \)

**Proposition:** Every monotone bounded sequence converges.

**Proof:** If \( \{a_n\} \) is monotone and bounded, then it is also bounded above. Let \( M \) be an upper bound for \( \{a_n\} \). Then, for each \( n \), \( a_n \leq M \).

Suppose \( \{a_n\} \) is non-decreasing. Then, \( a_n \leq a_{n+1} \) for all \( n \). Therefore, \( a_n \rightarrow a_{\infty} \).

**Example:**

- \( a_n = \frac{3n}{n+2} \)

**Function:**

- \( f(x) = \frac{3x}{x+2} \)

**Series:**

- \( a_n = \frac{3n}{n+2} \)