The Root Test

The root test for convergence or divergence of series works especially well for series involving nth powers. The proof of this theorem is similar to that given for the Ratio Test, and is left as an exercise (see Exercise 96).

**Theorem 9.18: Root Test**

Let \( \sum a_n \) be a series.
1. \( \sum a_n \) converges absolutely if \( \lim_{n \to \infty} |a_n|^{1/n} < 1 \).
2. \( \sum a_n \) diverges if \( \lim_{n \to \infty} |a_n|^{1/n} > 1 \) or \( \lim_{n \to \infty} |a_n|^{1/n} = \infty \).
3. The Root Test is inconclusive if \( \lim_{n \to \infty} |a_n|^{1/n} = 1 \).

**Example 4: Using the Root Test**

Determine the convergence or divergence of

\[
\sum \frac{1}{n^2 + n + 1}
\]

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges).
Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. Here is a summary in the table on page 644. Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the 2n-th term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

EXAMPLE 1 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

\[
\lim_{n \to \infty} \frac{3^n}{n^3} = \infty
\]

By Ratio Test
Our Series Diverges
85. **AH. Solved Test**

1) \( \lim_{n \to \infty} \frac{5}{n} = 0 \)

Converges

By AH. Solved Test

2) \( \frac{5}{n} \leq \frac{5}{n} \)

5 \leq 5

Always True

52. \( \frac{\frac{\sqrt{7}}{2}}{\frac{5}{2}} \)

\( \text{Rewrite:} \)

\( \beta = \frac{3}{2} \)

Convergent

55. \( \sum_{n=1}^{\infty} \frac{2n}{n+1} \)

\( \text{NTH Term Test} \)

\( \lim_{n \to \infty} \frac{2n}{n+1} = 2 \)

Diverges

By nth Term Test

59. \( \frac{\sqrt{3}}{\sqrt{x}} \)

\( \text{Rewrite:} \)

\( \lim_{x \to \infty} \frac{\sqrt{3}}{\sqrt{x}} = 0 \)

Convergent

By Root Test

\( \lim_{x \to \infty} \frac{\sqrt{3}}{\sqrt{x}} = 0 \)

Convergent by Root Test
8. \[ \lim_{n \to \infty} \frac{\ln n}{n^{1/2}} \] 

\[ \lim_{n \to \infty} \frac{\ln n}{n^{1/2}} = \frac{0}{0} \]

= 0

Converges since \( b = 0 \) is a finite real number.

By Limit Comparison Test.

9. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] 

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]

Convergent by Comparison Test to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

10. \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n} \]

By Direct Comparison Test, \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

\[ \frac{e^{1/n}}{n} \leq \frac{1}{n^2} \]

\( e^{1/n} \) is always true.

11. \[ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \]

\[ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \]

Convergent by Ratio Test.

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = 0 < 1 \]