Section 9.10

Binomial Theorem

Let $x = e^{i\theta}$

$$(1 + x)^n = 1 + \frac{1}{1!} \frac{n(n - 1)}{2!} \left( \frac{x}{k} \right)^2 + \frac{n(n - 1)(n - 2)}{3!} \left( \frac{x}{k} \right)^3 + \ldots$$

$-1 < x < 1$

* The convergence at $x = 1$ depends on the value of $k$.

\[ \frac{1}{2} \left[ 1 + \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^2 + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^3 \right] + \left( \frac{x}{k} \right)^4 + \ldots \]

\[ \frac{1}{2} \left[ 1 + \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^2 + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^3 \right] + \left( \frac{x}{k} \right)^4 + \ldots \]

\[ \frac{1}{2} \left[ 1 + \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^2 + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) \left( \frac{x}{k} \right)^3 \right] + \left( \frac{x}{k} \right)^4 + \ldots \]

\[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} \left( \frac{x}{k} \right)^{2n} \]

\[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot x^{2n}}{2^{3n+1} \cdot n!} \]
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \quad -\infty < x < \infty \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

Assum:
\[ \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = 1 \]
\[ 0 \]
\[ \cos x \]
\[ \frac{1}{1} \]

\[ \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right]_0^1 = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) \right] - 0 \]

\[ \text{Remainder for Alt. Series Test} \]
\[ |R_3| \leq \frac{\delta}{(2n+1)!} \]

Error < 0.001
\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \sim 0.966 \]

Area under curve /
In Exercises 53–56, match the polynomial with its graph. [The graphs are labeled (a), (b), (c), and (d).] Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.

53. \( y = x^2 - \frac{x^4}{2!} \)
54. \( y = x - \frac{x^3}{2!} + \frac{x^5}{4!} \)
55. \( y = x + x^2 + \frac{x^3}{2!} \)
56. \( y = x^2 - x^3 + x^4 \)

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots \]
Conics and Calculus

- Understand the definition of a conic section.
- Analyze and write equations of parabolas using properties of parabolas.
- Analyze and write equations of ellipses using properties of ellipses.
- Analyze and write equations of hyperbolas using properties of hyperbolas.

Conic Sections

Each conic section (or simply conic) can be described as the intersection of a plane and a double-napped cone. Notice in Figure 10.1 that for the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane passes through the vertex, the resulting figure is a degenerate conic, as shown in Figure 10.2.

![Conic Sections Diagram](Image)

There are several ways to study conics. You could begin as the Greeks did by defining the conics in terms of the intersections of planes and cones, or you could define them algebraically in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$  
General second-degree equation

However, a third approach, in which each of the conics is defined as a locus (collection) of points satisfying a certain geometric property, works best. For example, a circle can be defined as the collection of all points \((x, y)\) that are equidistant from a fixed point \((h, k)\). This locus definition easily produces the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$  
Standard equation of a circle

For information about rotating second-degree equations in two variables, see Appendix D.
Parabolas

A parabola is the set of all points \((x, y)\) that are equidistant from a fixed line called the directrix and a fixed point called the focus not on the line. The midpoint between the focus and the directrix is the vertex, and the line passing through the focus and the vertex is the axis of the parabola. Note in Figure 10.3 that a parabola is symmetric with respect to its axis.

**THEOREM 10.1 STANDARD EQUATION OF A PARABOLA**

The standard form of the equation of a parabola with vertex \((h, k)\) and directrix \(y = k - p\) is

\[
(x - h)^2 = 4p(y - k).
\]

Vertical axis

For directrix \(x = h - p\), the equation is

\[
(y - k)^2 = 4p(x - h).
\]

Horizontal axis

The focus lies on the axis \(p\) units (directed distance) from the vertex. The coordinates of the focus are as follows.

\[ (h, k + p) \quad \text{Vertical axis} \]

\[ (h + p, k) \quad \text{Horizontal axis} \]

**Figure 10.3**

\[ y = k - p \quad \text{Directrix line} \]
Try It 1

Find the focus of the parabola given by \( y = -\frac{1}{6}(x^2 + 4x - 2) \).

The standard form of the equation directrix \( y = k - p \) is

\[
(x - h)^2 = 4p(y - k).
\]

Complete Square

\[
y = -\frac{1}{6}x^2 - \frac{2}{3}x + \frac{2}{6}
\]

Factor Out \(-\frac{1}{6}\)

\[-6y = x^2 + 4x - 2\]

Clear Fraction

Complete Square

\[
-6y = (x+2)^2 - 4\]

Add 6

\[
\frac{-6y + 6}{-6} = (x+2)^2
\]

\[
-(y-1) = (x+2)^2
\]

Standard Form

Vertex

\( (h, k) \) \(( -2, 1)\)

Focus

\( p = -\frac{3}{2} \)

Directrix Line

\( y = \frac{5}{2} \)
A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord**. The specific focal chord perpendicular to the axis of the parabola is the **latus rectum**. The next example shows how to determine the length of the latus rectum and the length of the corresponding intercepted arc.

![Graph](image)

Length of latus rectum: $4p$
Figure 10.5

**Example 2** Focal Chord Length and Arc Length

Find the length of the latus rectum of the parabola given by $x^2 = 4py$. Then find the length of the parabolic arc intercepted by the latus rectum.

**Solution** Because the latus rectum passes through the focus $(0, p)$ and is perpendicular to the $y$-axis, the coordinates of its endpoints are $(-x, p)$ and $(x, p)$. Substituting $p$ for $y$ in the equation of the parabola produces

$$x^2 = 4p^2(p)$$

$x = \pm 2p$.

So, the endpoints of the latus rectum are $(-2p, p)$ and $(2p, p)$, and you can conclude that its length is $4p$, as shown in Figure 10.5. In contrast, the length of the intercepted arc is

\[
s = \int_{-2p}^{2p} \sqrt{1 + (\frac{x}{2p})^2} \, dx
\]

\[
= \frac{1}{p} \int_{-2p}^{2p} \sqrt{4p^2 + x^2} \, dx
\]

\[
= \frac{1}{2p} \left[ x \sqrt{4p^2 + x^2} + 4p^2 \ln|x + \sqrt{4p^2 + x^2}| \right]_0^{2p}
\]

\[
= \frac{1}{2p} \left[ 2p \sqrt{8p^2} + 4p^2 \ln(2p + \sqrt{8p^2}) - 4p^2 \ln(2p) \right]
\]

\[
= 2p \left[ \sqrt{2} + \ln(1 + \sqrt{2}) \right]
\]

\[
= 4.59p.
\]

Use arc length formula.

$y = \frac{x^2}{4p} \quad \Rightarrow \quad y' = \frac{x}{2p}$

Simplify.

Theorem 8.2
One widely used property of a parabola is its reflective property. In physics, a surface is called reflective if the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the *angle of incidence*, and the angle corresponding to the outgoing ray is the *angle of reflection*. One example of a reflective surface is a flat mirror.

Another type of reflective surface is that formed by revolving a parabola about its axis. A special property of parabolic reflectors is that they allow us to direct all incoming rays parallel to the axis through the focus of the parabola—this is the principle behind the design of the parabolic mirrors used in reflecting telescopes. Conversely, all light rays emanating from the focus of a parabolic reflector used in a flashlight are parallel, as shown in Figure 10.6.

**Theorem 10.2 Reflective Property of a Parabola**

Let $P$ be a point on a parabola. The tangent line to the parabola at the point $P$ makes equal angles with the following two lines.

1. The line passing through $P$ and the focus
2. The line passing through $P$ parallel to the axis of the parabola

---

**Figure 10.6**

Parabolic reflector: light is reflected in parallel rays.
Ellipses

More than a thousand years after the close of the Alexandrian period of Greek mathematics, Western civilization finally began a Renaissance of mathematical and scientific discovery. One of the principal figures in this rebirth was the Polish astronomer Nicolaus Copernicus. In his work On the Revolutions of the Heavenly Spheres, Copernicus claimed that all of the planets, including Earth, revolved about the sun in circular orbits. Although some of Copernicus's claims were invalid, the controversy set off by his heliocentric theory motivated astronomers to search for a mathematical model to explain the observed movements of the sun and planets. The first to find an accurate model was the German astronomer Johannes Kepler (1571–1630). Kepler discovered that the planets move about the sun in elliptical orbits, with the sun not as the center but as a focal point of the orbit.

The use of ellipses to explain the movements of the planets is only one of many practical and aesthetic uses. As with parabolas, you will begin your study of this second type of conic by defining it as a locus of points. Now, however, two focal points are used rather than one.

An ellipse is the set of all points \((x, y)\) the sum of whose distances from two distinct fixed points called foci is constant. (See Figure 10.7.) The line through the foci intersects the ellipse at two points, called the vertices. The chord joining the vertices is the major axis, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the minor axis of the ellipse. (See Figure 10.8.)

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FOR FURTHER INFORMATION  To learn about how an ellipse may be “exploded” into a parabola, see the article “Exploding the Ellipse” by Arnold Good in Mathematics Teacher. To view this article, go to the website www.matharticles.com.

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**THEOREM 10.3 STANDARD EQUATION OF AN ELLIPSE**

The standard form of the equation of an ellipse with center \((h, k)\) and major and minor axes of lengths \(2a\) and \(2b\), where \(a > b\), is

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1
\]

Major axis is horizontal.

or

\[
\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1.
\]

Major axis is vertical.

The foci lie on the major axis, \(c\) units from the center, with \(c^2 = a^2 - b^2\).

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.9.
Try It 3

Find the center, vertices, and foci of the ellipse given by

$$16x^2 + 25y^2 - 32x + 50y + 31 = 0.$$  

The graph of the ellipse is shown in the figure.

Center \((1, -1)\)

$$a^2 = \frac{5}{3}$$

$$a = \sqrt{\frac{5}{3}}$$

Vertices \(\left(1 \pm \sqrt{\frac{5}{3}}, -1\right)\) \(\left(1, 1 \pm \sqrt{\frac{5}{3}}\right)\)

Foci \(\left(1 \pm \frac{3}{4\sqrt{5}}, -1\right)\)

Complete Square

$$\frac{16(x^2 - 2x + 1)}{16} + \frac{25(y^2 + 2y + 1)}{25} = -31 + \frac{16}{16} + \frac{25}{25}$$

$$\frac{(x-1)^2}{\frac{5}{8}} + \frac{(y+1)^2}{\frac{2}{5}} = 1$$

Standard Form

$$\frac{(x-1)^2}{\frac{5}{8}} + \frac{(y+1)^2}{\frac{2}{5}} = 1$$

$$c^2 = \frac{5}{8} - \frac{2}{5} = \frac{9}{40}$$

$$c = \frac{3}{4\sqrt{5}}$$
**NOTE** If the constant term $F = -8$ in the equation in Example 3 had been greater than or equal to 8, you would have obtained one of the following degenerate cases.

1. $F = 8$, single point, $(1, -2)$: \( \frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} = 0 \)

2. $F > 8$, no solution points: \( \frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} < 0 \)

**EXAMPLE 4** The Orbit of the Moon

The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in Figure 10.11. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from Earth’s center to the moon’s center.

**Solution** Begin by solving for $a$ and $b$.

\[
2a = 768,800 \quad \text{Length of major axis}
\]

\[
a = 384,400 \quad \text{Solve for } a.
\]

\[
2b = 767,640 \quad \text{Length of minor axis}
\]

\[
b = 383,820 \quad \text{Solve for } b.
\]

Now, using these values, you can solve for $c$ as follows.

\[
c = \sqrt{a^2 - b^2} = 21,108
\]

The greatest distance between the center of Earth and the center of the moon is $a + c = 405,508$ kilometers, and the least distance is $a - c = 363,292$ kilometers.
Theorem 10.2 presented a reflective property of parabolas. Ellipses have a similar reflective property. You are asked to prove the following theorem in Exercise 112.

**THEOREM 10.4 REFLECTIVE PROPERTY OF AN ELLIPSE**

Let \( P \) be a point on an ellipse. The tangent line to the ellipse at point \( P \) makes equal angles with the lines through \( P \) and the foci.

One of the reasons that astronomers had difficulty in detecting that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to the center of the sun, making the orbits nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity**.

**DEFINITION OF ECCENTRICITY OF AN ELLIPSE**

The eccentricity \( e \) of an ellipse is given by the ratio

\[
e = \frac{c}{a}
\]

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

\[ 0 < c < a. \]

For an ellipse that is nearly circular, the foci are close to the center and the ratio \( c/a \) is small, and for an elongated ellipse, the foci are close to the vertices and the ratio \( c/a \) is close to 1, as shown in Figure 10.12. Note that \( 0 < e < 1 \) for every ellipse.

The orbit of the moon has an eccentricity of \( e = 0.0549 \), and the eccentricities of the eight planetary orbits are as follows.

- Mercury: \( e = 0.2056 \)
- Jupiter: \( e = 0.0484 \)
- Venus: \( e = 0.0068 \)
- Saturn: \( e = 0.0542 \)
- Earth: \( e = 0.0167 \)
- Uranus: \( e = 0.0472 \)
- Mars: \( e = 0.0934 \)
- Neptune: \( e = 0.0086 \)

You can use integration to show that the area of an ellipse is \( A = \pi ab \). For instance, the area of the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

is given by

\[
A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx
\]

\[
= \frac{4b}{a} \int_0^{\pi/2} a^2 \cos^2 \theta \, d\theta.
\]

**Trigonometric substitution**

\[
s = a \sin \theta
\]

However, it is not so simple to find the **circumference** of an ellipse. The next example shows how to use eccentricity to set up an “elliptic integral” for the circumference of an ellipse.
EXAMPLE 5 Finding the Circumference of an Ellipse

Show that the circumference of the ellipse \((x^2/a^2) + (y^2/b^2) = 1\) is

\[
4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \, d\theta. \quad e = \frac{c}{a}
\]

**Solution** Because the given ellipse is symmetric with respect to both the x-axis and the y-axis, you know that its circumference \(C\) is four times the arc length of \(y = (b/a) \sqrt{a^2 - x^2}\) in the first quadrant. The function \(y\) is differentiable for all \(x\) in the interval \([0, a]\) except at \(x = a\). So, the circumference is given by the improper integral

\[
C = \lim_{d \to a} 4 \int_0^d \sqrt{1 + (y')^2} \, dx = 4 \int_0^a \sqrt{1 + (y')^2} \, dx = 4 \int_0^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} \, dx.
\]

Using the trigonometric substitution \(x = a \sin \theta\), you obtain

\[
C = 4 \int_0^{\pi/2} \sqrt{1 + \frac{b^2 \sin^2 \theta}{a^2 \cos^2 \theta}} (a \cos \theta) \, d\theta
\]

\[
= 4 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta
\]

\[
= 4 \int_0^{\pi/2} \sqrt{a^2 (1 - \sin^2 \theta) + b^2 \sin^2 \theta} \, d\theta
\]

\[
= 4 \int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2) \sin^2 \theta} \, d\theta.
\]

Because \(e^2 = c^2/a^2 = (a^2 - b^2)/a^2\), you can rewrite this integral as

\[
C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \, d\theta.
\]

A great deal of time has been devoted to the study of elliptic integrals. Such integrals generally do not have elementary antiderivatives. To find the circumference of an ellipse, you must usually resort to an approximation technique.
Try It 6

Use the elliptic integral to approximate the circumference of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$ 

Solution

Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2 = 7/16$, you have

$$C = (4)(4) \int_0^{\pi/2} \sqrt{1 - \frac{7}{16} \sin^2 \theta} \ d\theta.$$ 

Applying Simpson’s Rule with $n = 12$ produces $c \approx 22.10$.
So, the ellipse has a circumference of about 22.10 units.

Applying Simpson’s Rule with $n = 12$ produces $c \approx 22.10$.
So, the ellipse has a circumference of about 22.10 units.
Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the sum of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the difference between these distances is fixed.

A hyperbola is the set of all points \((x, y)\) for which the absolute value of the difference between the distances from two distinct fixed points called foci is constant. (See Figure 10.14.) The line through the two foci intersects a hyperbola at two points called the vertices. The line segment connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola. One distinguishing feature of a hyperbola is that its graph has two separate branches.

**Theorem 10.5 Standard Equation of a Hyperbola**

The standard form of the equation of a hyperbola with center at \((h, k)\) is

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}
\]

or

\[
\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}
\]

The vertices are \(a\) units from the center, and the foci are \(c\) units from the center, where \(c^2 = a^2 + b^2\).

**Note** The constants \(a, b,\) and \(c\) do not have the same relationship for hyperbolas as they do for ellipses. For hyperbolas, \(c^2 = a^2 + b^2\), but for ellipses, \(c^2 = a^2 - b^2\).

**Theorem 10.6 Asymptotes of a Hyperbola**

For a horizontal transverse axis, the equations of the asymptotes are

\[
y = k + \frac{b}{a}(x - h) \quad \text{and} \quad y = k - \frac{b}{a}(x - h).
\]

For a vertical transverse axis, the equations of the asymptotes are

\[
y = k + \frac{a}{b}(x - h) \quad \text{and} \quad y = k - \frac{a}{b}(x - h).
\]

In Figure 10.15 you can see that the asymptotes coincide with the diagonals of the rectangle with dimensions \(2a\) and \(2b\), centered at \((h, k)\). This provides you with a quick means of sketching the asymptotes, which in turn aids in sketching the hyperbola.
Try It 7

Sketch the graph of the hyperbola whose equation is $x^2 - 9y^2 = 36$.

Positive

Center $(0, 0)$

$b = \frac{2}{a} = \sqrt{6}$

Slope Asymptotes

$c^2 = 36 + 4 = 40$

$c = \sqrt{40}$

Figure 1

Figure 2
DEFINITION OF ECCENTRICITY OF A HYPERBOLA

The eccentricity $e$ of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

As with an ellipse, the eccentricity of a hyperbola is $e = c/a$. Because $c > a$ for hyperbolas, it follows that $e > 1$ for hyperbolas. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 10.17.
Try It 8

Two microphones, 3 miles apart, record an explosion. Microphone A receives the sound 5 seconds before microphone B. Where was the explosion?

Solution

Assuming that sound travels at 1100 feet per second, you know that the explosion took place 5500 feet farther from B than from A, as shown in the figure. The locus of all points that are 5500 feet closer to A than to B is one branch of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, where

$$c = \frac{\text{3 miles}}{2} = \frac{15,840 \text{ ft}}{2} = 7920 \text{ ft}$$

and

$$a = \frac{5500 \text{ ft}}{2} = 2750 \text{ ft}.$$ 

Because $c^2 = a^2 + b^2$, it follows that

$$b^2 = c^2 - a^2$$

$$= 55,163,900$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola given by

$$\frac{x^2}{7,562,500} - \frac{y^2}{55,163,900} = 1.$$
In Example 8, you were able to determine only the hyperbola on which the explosion occurred, but not the exact location of the explosion. If, however, you had received the sound at a third position \( C \), then two other hyperbolas would be determined. The exact location of the explosion would be the point at which these three hyperbolas intersect.

Another interesting application of conics involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each orbit, and each orbit has a vertex at the point at which the comet is closest to the sun. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified—such comets pass through our solar system only once. Only comets with elliptical orbits such as Halley’s comet remain in our solar system.

The type of orbit for a comet can be determined as follows.

1. Ellipse: \[ v < \sqrt{2GM/p} \]
2. Parabola: \[ v = \sqrt{2GM/p} \]
3. Hyperbola: \[ v > \sqrt{2GM/p} \]

In these three formulas, \( p \) is the distance between one vertex and one focus of the comet’s orbit (in meters), \( v \) is the velocity of the comet at the vertex (in meters per second), \( M \approx 1.989 \times 10^{30} \) kilograms is the mass of the sun, and \( G \approx 6.67 \times 10^{-8} \) cubic meters per kilogram-second squared is the gravitational constant.
10.1 Exercises

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

(a) \[ y^2 = 4x \]
(b) \[ y^2 = x + y \]
(c) \[ x^2 + 4x - 6 = 0 \]
(d) \[ x^2 + y^2 = 4 \]
(e) \[ y^2 = 4x \]
(f) \[ x^2 + y^2 = 4 \]
(g) \[ x^2 + y^2 = 4 \]
(h) \[ x^2 + y^2 = 4 \]

1. \[ y^2 = 4x \]
2. \[ (x + 4)^2 = 2(y + 2) \]
3. \[ (x + 4)^2 = -2(y - 2) \]
4. \[ (x - 2)^2 + (y + 1)^2 = 1 \]
5. \[ x^2 + y^2 = 1 \]
6. \[ x^2 + y^2 = 1 \]
7. \[ x^2 + y^2 = 1 \]
8. \[ (x - 2)^2 - y^2 = 1 \]

In Exercises 9–16, find the vertex, focus, and directrix of the parabola, and sketch its graph.

9. \[ y^2 = -8x \]
10. \[ x^2 + 6y = 0 \]
11. \[ (x + 5) + (y - 3)^2 = 0 \]
12. \[ (x - 6)^2 + 8(y + 7) = 0 \]
13. \[ y^2 - 4y - 4x = 0 \]
14. \[ y^2 + 6y + 8x + 25 = 0 \]
15. \[ x^2 + 4x + 4y - 4 = 0 \]
16. \[ x^2 + 4y + 8x - 12 = 0 \]

In Exercises 17–20, find the vertex, focus, and directrix of the parabola. Then use a graphing utility to graph the parabola.

17. \[ y^2 + x + y = 0 \]
18. \[ y = -\frac{2}{3}(x^2 - 8x + 6) \]
19. \[ y^2 - 4x - 4 = 0 \]
20. \[ x^2 - 2x + 8y + 9 = 0 \]

In Exercises 21–28, find an equation of the parabola.

21. Vertex: \((5, 4)\), Focus: \((3, 4)\)
22. Vertex: \((-2, 1)\), Focus: \((-2, -1)\)
23. Vertex: \((0, 5)\), Directrix: \(y = -3\)
24. Focus: \((2, 2)\), Directrix: \(x = -2\)

25. \[ (t - 2, 0) \]
26. \[ (2, 4) \]
27. Axis is parallel to \(y\)-axis; graph passes through \((0, 3), (3, 4),\) and \((4, 11)\).
28. Directrix: \(y = -2\); endpoints of latus rectum are \((0, 2)\) and \((8, 2)\).

In Exercises 29–34, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.

29. \[ 16x^2 + y^2 = 16 \]
30. \[ 3x^2 + 7y^2 = 63 \]
31. \[ \frac{(x - 3)^2}{16} + \frac{(y - 1)^2}{25} = 1 \]
32. \[ (x + 4)^2 + \frac{(y + 3)^2}{1/4} = 1 \]
33. \[ 9x^2 + 4y^2 + 36x - 24y + 36 = 0 \]
34. \[ 16x^2 + 25y^2 - 64x + 150y + 279 = 0 \]

In Exercises 35–38, find the center, foci, and vertices of the ellipse. Use a graphing utility to graph the ellipse.

35. \[ 12x^2 + 20y^2 - 12x + 40y - 37 = 0 \]
36. \[ 36x^2 + 9y^2 + 48x - 30y + 43 = 0 \]
37. \[ x^2 + 2y^2 - 3x + 4y + 0.25 = 0 \]
38. \[ 2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0 \]

In Exercises 39–44, find an equation of the ellipse.

39. Center: \((0, 0)\)
   Focus: \((5, 0)\)
   Vert: \((6, 0)\)
40. Vertices: \((0, 3), (8, 3)\)
   Eccentricity: \(\frac{1}{2}\)
41. Vert: \((3, 1), (3, 9)\)
   Minor axis length: 6
42. Foci: \((0, -9)\)
   Major axis length: 22
43. Center: \((0, 0)\)
   Major axis: vertical
   Points on the ellipse: \((3, 1), (4, 0)\)
44. Center: \((1, 2)\)
   Major axis: horizontal
   Points on the ellipse: \((1, 6), (3, 2)\)
In Exercises 45–52, find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.

45. \( \frac{x^2}{9} - \frac{y^2}{9} = 1 \)  
46. \( \frac{x^2}{25} - \frac{y^2}{16} = 1 \)  
47. \( \frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1 \)  
48. \( \frac{(x+3)^2}{225} - \frac{(y-5)^2}{64} = 1 \)  
49. \( 9x^2 - y^2 - 36x - 6y + 8 = 0 \)  
50. \( y^2 - 10x^2 + 64y = 208 \)  
51. \( x^2 - 9y^2 + 2x - 54y = 80 \)  
52. \( 9x^2 - 4y^2 + 54x + 8y + 78 = 0 \)

In Exercises 53–56, find the center, foci, and vertices of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

53. \( 9y^2 + 2x + 54y + 62 = 0 \)  
54. \( 9x^2 + 4y^2 + 10y + 55 = 0 \)  
55. \( 3x^2 - 2y^2 - 6x - 12y = 27 \)  
56. \( 3y^2 - x^2 + 6x - 12y = 0 \)

In Exercises 57–64, find an equation of the hyperbola.

57. Vertices: \((\pm 1, 0)\)  
   Asymptotes: \(y = \pm 5x\)  
58. Vertices: \((0, \pm 4)\)  
   Asymptotes: \(y = \pm 2x\)  
59. Vertices: \((2, \pm 3)\)  
   Foci: \((2, \pm 5)\)  
60. Vertices: \((2, \pm 3)\)  
   Foci: \((0, 4)\)

61. Center: \((0, 0)\)  
   Vertices: \((0, 2)\)  
   Foci: \((0, 4)\)  
   Focus: \((10, 0)\)  
62. Center: \((0, 0)\)  
   Vertices: \((0, 2)\)  
   Foci: \((20, 0)\)

In Exercises 65 and 66, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of \(x\).

65. \( \frac{x^2}{9} - y^2 = 1 \)  
   (a) \( x = 6 \)  
   (b) \( x = 4 \)

66. \( \frac{x^2}{4} - \frac{y^2}{2} = 1 \)  
   (a) \( x = 4 \)  
   (b) \( x = 3 \)

In Exercises 67–76, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

67. \( x^2 + 4y^2 - 6x + 16y + 21 = 0 \)  
68. \( 4x^2 - y^2 - 4x = 0 \)  
69. \( x^2 - 8y - 8x = 0 \)  
70. \( 25x^2 - 10x - 200y - 119 = 0 \)  
71. \( 4x^2 + 4y^2 - 16y + 15 = 0 \)  
72. \( y^2 - 4y = x + 5 \)  
73. \( 9x^2 + 9y^2 - 36x + 6y + 34 = 0 \)  
74. \( 2x(x-y) = y(3y - 2x) \)  
75. \( 3(x - 1)^2 = 6 + 2(y + 1)^2 \)  
76. \( 9(x + 3)^2 = 36 - 4(y - 2)^2 \)

81. Solar Collector A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

82. Beam Deflection A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.
   (a) Find the equation of the parabola. (Assume that the origin is at the center of the beam.)
   (b) How far from the center of the beam is the deflection 1 centimeter?
86. Find the point on the graph of \( x^2 = 8y \) that is closest to the focus of the parabola.

87. Radio and Television Reception In mountainous areas, reception of radio and television is sometimes poor. Consider an idealized case where a hill is represented by the graph of the parabola \( y = x - x^2 \), a transmitter is located at the point \((-1, 1)\), and a receiver is located on the other side of the hill at the point \((x_t, 0)\). What is the closest the receiver can be to the hill while still maintaining unobstructed reception?

88. Modeling Data The table shows the average amounts of time \( A \) (in minutes) women spent watching television each day for the years 1999 through 2005. \( \text{(Source: Nielsen Media Research)} \)

<table>
<thead>
<tr>
<th>Year</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>280</td>
<td>286</td>
<td>291</td>
<td>298</td>
<td>305</td>
<td>307</td>
<td>317</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find a model of the form \( A = ax^2 + bx + c \) for the data. Let \( t \) represent the year, with \( t = 9 \) corresponding to 1999.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find \( dA/dt \) and sketch its graph for \( 9 \leq t \leq 15 \). What information about the average amount of time women spent watching television is given by the graph of the derivative?

89. Architecture A church window is bounded above by a parabola and below by the arc of a circle (see figure). Find the surface area of the window.

90. Arc Length Find the arc length of the parabola \( 4x - y^2 = 0 \) over the interval \( 0 \leq y \leq 4 \).

91. Bridge Design A cable of a suspension bridge is suspended (in the shape of a parabola) between two towers that are 120 meters apart and 20 meters above the roadway (see figure). The cables touch the roadway midway between the towers.

(a) Find an equation for the parabolic shape of each cable.

(b) Find the length of the parabolic supporting cable.

92. Surface Area A satellite signal receiving dish is formed by revolving the parabola given by \( y^2 = 20x \) about the \( y \)-axis. The radius of the dish is \( r \) feet. Verify that the surface area of the dish is given by

\[
2\pi \int_0^r x \sqrt{1 + \left( \frac{x}{10} \right)^2} \, dx = \frac{\pi}{15} \left[ 100 + r^2 \right]^{1/2} - 1000.
\]

93. Investigation Sketch the graphs of \( x^2 = 4py \) for \( p = \frac{1}{4}, \frac{1}{2}, 1, 2 \), and 2 on the same coordinate axes. Discuss the change in the graphs as \( p \) increases.

94. Area Find a formula for the area of the shaded region in the figure.

95. Writing On page 699, it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the tacks), and a pencil. If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

(a) What is the length of the string in terms of \( a \)?

(b) Explain why the path is an ellipse.

96. Construction of a Semielliptical Arch A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Exercise 95. Where should the tacks be placed and what should be the length of the piece of string?

97. Sketch the ellipse that consists of all points \((x, y)\) such that the sum of the distances between \((x, y)\) and two fixed points is 16 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

98. Orbit of Earth Earth moves in an elliptical orbit with the sun at one of the foci. The length of half of the major axis is 149,598,000 kilometers, and the eccentricity is 0.0167. Find the minimum distance (perihelion) and the maximum distance (aphelion) of Earth from the sun.

99. Satellite Orbit The apogee (the point in orbit furthest from Earth) and the perigee (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by \( A \) and \( P \). Show that the eccentricity of the orbit is

\[
e = \frac{A - P}{A + P}.
\]

100. Explorer 18 On November 27, 1963, the United States launched the research satellite Explorer 18. Its low and high points above the surface of Earth were 119 miles and 123,000 miles. Find the eccentricity of its elliptical orbit.
101. **Explorer 55** On November 20, 1975, the United States launched the research satellite Explorer 55. Its low and high points above the surface of Earth were 96 miles and 1865 miles. Find the eccentricity of its elliptical orbit.

**CAPSTONE**

102. Consider the equation

\[9x^2 + 4y^2 - 36x - 24y - 36 = 0.\]

(a) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

(b) Change the \(4y^2\)-term in the equation to \(-4y^2\). Classify the graph of the new equation.

(c) Change the \(9x^2\)-term in the original equation to \(4x^2\). Classify the graph of the new equation.

(d) Describe one way you could change the original equation so that its graph is a parabola.

103. **Halley’s Comet** Probably the most famous of all comets, Halley’s comet, has an elliptical orbit with the sun at one focus. Its maximum distance from the sun is approximately 35.29 AU (1 astronomical unit = 92,956 \times 10^6 miles), and its minimum distance is approximately 0.59 AU. Find the eccentricity of the orbit.

104. The equation of an ellipse with its center at the origin can be written as

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.\]

Show that as \(e \to 0\), with \(a\) remaining fixed, the ellipse approaches a circle.

105. Consider a particle traveling clockwise on the elliptical path

\[\frac{x^2}{100} + \frac{y^2}{25} = 1.\]

The particle leaves the orbit at the point \((-8, 3)\) and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

106. **Volume** The water tank on a fire truck is 16 feet long, and its cross sections are ellipses. Find the volume of water in the partially filled tank as shown in the figure.

In Exercises 107 and 108, determine the points at which \(dy/dx\) is zero or does not exist to locate the endpoints of the major and minor axes of the ellipse.

107. \(16x^2 + 9y^2 + 96x + 36y + 36 = 0\)

108. \(9x^2 + 4y^2 + 36x - 24y + 36 = 0\)

**Area and Volume** In Exercises 109 and 110, find (a) the area of the region bounded by the ellipse, (b) the volume and surface area of the solid generated by revolving the region about its major axis (prolate spheroid), and (c) the volume and surface area of the solid generated by revolving the region about its minor axis (oblate spheroid).

109. \(\frac{x^2}{4} + \frac{y^2}{1} = 1\)

110. \(\frac{x^2}{16} + \frac{y^2}{9} = 1\)

111. **Arc Length** Use the integration capabilities of a graphing utility to approximate to two-decimal-place accuracy the elliptical integral representing the circumference of the ellipse

\[\frac{x^2}{25} + \frac{y^2}{49} = 1.\]

112. Prove Theorem 10.4 by showing that the tangent line to an ellipse at a point \(P\) makes equal angles with lines through \(P\) and the foci (see figure). [**Hint:** (1) Find the slope of the tangent line at \(P\), (2) find the slopes of the lines through \(P\) and each focus, and (3) use the formula for the tangent of the angle between two lines.]

113. **Geometry** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis?

114. **Conjecture**

(a) Show that the equation of an ellipse can be written as

\[\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.\]

(b) Use a graphing utility to graph the ellipse

\[\frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{4(1 - e^2)} = 1\]

for \(e = 0.95, e = 0.75, e = 0.5, e = 0.25,\) and \(e = 0.\)

(c) Use the results of part (b) to make a conjecture about the change in the shape of the ellipse as \(e\) approaches 0.

115. Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points \( (2, 2) \) and \((10, 2) \) is 6.

116. Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points \((-3, 0)\) and \((-3, 3)\) is 2.
117. Sketch the hyperbola that consists of all points \((x, y)\) such that the difference of the distances between \((x, y)\) and two fixed points is 10 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

119. **Sound Location** A rifle positioned at point \((-c, 0)\) is fired at a target positioned at point \((c, 0)\). A person hears the sound of the rifle and the sound of the bullet hitting the target at the same time. Prove that the person is positioned on one branch of the hyperbola given by

\[
\frac{x^2}{c^2} - \frac{y^2}{v_m^2} = 1
\]

where \(v_m\) is the muzzle velocity of the rifle and \(v_s\) is the speed of sound, which is about 1100 feet per second.

120. **Navigation** LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (180,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at \((-150, 0)\) and \((150, 0)\) and that a ship is traveling on a path with coordinates \((x, y)\) (see figure). Find the \(x\)-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

121. **Hyperbolic Mirror** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The mirror in the figure has the equation \((x^2/36) - (y^2/64) = 1\). At which point on the mirror will light from the point \((0, 10)\) be reflected to the other focus?

122. Show that the equation of the tangent line to \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) at the point \((x_0, y_0)\) is \((x_0/a^2)x - (y_0/b^2)y = 1\).

123. Show that the graphs of the equations intersect at right angles:

\[
\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 1.
\]

124. Prove that the graph of the equation

\[Ax^2 + Cy^2 + Dx + Ey + F = 0\]

is one of the following (except in degenerate cases):

<table>
<thead>
<tr>
<th>Conic</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Circle</td>
<td>(A = C)</td>
</tr>
<tr>
<td>(b) Parabola</td>
<td>(A = 0) or (C = 0) (but not both)</td>
</tr>
<tr>
<td>(c) Ellipse</td>
<td>(AC &gt; 0)</td>
</tr>
<tr>
<td>(d) Hyperbola</td>
<td>(AC &lt; 0)</td>
</tr>
</tbody>
</table>

**True or False?** In Exercises 125–130, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

125. It is possible for a parabola to intersect its directrix.

126. The point on a parabola closest to its focus is its vertex.

127. If \(C\) is the circumference of the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b < a
\]

then \(2\pi b \leq C \leq 2\pi a\).

128. If \(D \neq 0\) or \(E \neq 0\), then the graph of \(y^2 - x^2 + Dx + Ey = 0\) is a hyperbola.

129. If the asymptotes of the hyperbola \((x^2/a^2) - (y^2/b^2) = 1\) intersect at right angles, then \(a = b\).

130. Every tangent line to a hyperbola intersects the hyperbola only at the point of tangency.

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**PUTNAM EXAM CHALLENGE**

131. For a point \(P\) on an ellipse, let \(d\) be the distance from the center of the ellipse to the line tangent to the ellipse at \(P\). Prove that \((PF_1)(PF_2)d\) is constant as \(P\) varies on the ellipse, where \(PF_1\) and \(PF_2\) are the distances from \(P\) to the foci \(F_1\) and \(F_2\) of the ellipse.

132. Find the minimum value of \((u - v)^2 + \left(\sqrt{2} - u - v\right)^2\)

for \(0 < u < \sqrt{2}\) and \(v > 0\).

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