Slope and Tangent Lines

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. To begin, let’s take another look at the projectile represented by the parametric equations

\[ x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t \]

as shown in Figure 10.29. From the discussion at the beginning of Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of 45°. But how can you find the angle \( \theta \) representing the object’s direction at some other time \( t \)? The following theorem answers this question by giving a formula for the slope of the tangent line as a function of \( t \).

**THEOREM 10.7 PARAMETRIC FORM OF THE DERIVATIVE**

If a smooth curve \( C \) is given by the equations \( x = f(t) \) and \( y = g(t) \), then the slope of \( C \) at \((x, y)\) is

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0. \]

**Proof** In Figure 10.30, consider \( \Delta t > 0 \) and let

\[ \Delta y = g(t + \Delta t) - g(t) \quad \text{and} \quad \Delta x = f(t + \Delta t) - f(t). \]

Because \( \Delta x \to 0 \) as \( \Delta t \to 0 \), you can write

\[ \frac{dy}{dx} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} \]

\[ = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}. \]

Dividing both the numerator and denominator by \( \Delta t \), you can use the differentiability of \( f \) and \( g \) to conclude that
Try It 1

Find \( \frac{dy}{dx} \) for the curve given by \( x = e^t \) and \( y = e^{-t} \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t}}{e^t}
\]

\[
\frac{dy}{dx} = \frac{-1}{e^{2t}}
\]

\[
\left. \frac{dy}{dx} \right|_{t=1} = -\frac{1}{e^2}
\]

\[
\text{Slope at } t=1 = -\frac{1}{e^2}
\]
Because $dy/dx$ is a function of $t$, you can use Theorem 10.7 repeatedly to find higher-order derivatives. For instance,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ \frac{dy}{dx} \right] \frac{dx}{dt}.$$  

**Second derivative**

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left[ \frac{d^2y}{dx^2} \right] = \frac{d}{dt} \left[ \frac{d^2y}{dx^2} \right] \frac{dx}{dt}.$$  

**Third derivative**
Try It 2

For the curve given by

\[ x = t^2 + 3t \quad \text{and} \quad y = t + 1 \]

find the slope and concavity at the point (0, 1).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t+3}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{\frac{d}{dt} \left[ \frac{dy}{dx} \right]}{\frac{dx}{dt}}
\]

\[
\left. \frac{d^2y}{dx^2} \right|_{t=0} = \frac{1}{3}
\]

Concave Down
Because the parametric equations \( x = f(t) \) and \( y = g(t) \) need not define \( y \) as a function of \( x \), it is possible for a plane curve to loop around and cross itself. At such points the curve may have more than one tangent line, as shown in the next example.

**Try It 3**

The prolate cycloid given by
\[
\begin{align*}
x &= t^2 - t + 2 \\
y &= t^3 - 3t
\end{align*}
\]
crosses itself at the point \((4, 2)\), as shown in the figure. Find the equations of both tangent lines at this point.

**Horizontal Tangent**

- **Slope = 0** happens when \( \frac{dy}{dt} = 0 \)

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t - 1}
\]

\( t = 1 \):

\[
\frac{dy}{dx} = \frac{3(1)^2 - 3}{2(1) - 1} = \frac{0}{1} = 0
\]

**Vertical Tangent**

- **Slope Undefined** happens when \( \frac{dx}{dt} = 0 \)

\[
3t^2 - 3 = 0
\]

\( t = \pm 1 \)

\[
3(t^2 - 1) = 0
\]

\( t = 1, t = -1 \)

\[
2t - 1 = 0
\]

\( t = \frac{1}{2} \)
**Arc Length**

You have seen how parametric equations can be used to describe the path of a particle moving in the plane. You will now develop a formula for determining the distance traveled by the particle along its path.

Recall from Section 7.4 that the formula for the arc length of a curve \( C \) given by \( y = h(x) \) over the interval \([x_0, x_1]\) is

\[
s = \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2} \, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]

If \( C \) is represented by the parametric equations \( x = f(t) \) and \( y = g(t) \), \( a \leq t \leq b \), and if \( dx/dt = f'(t) > 0 \), you can write

\[
s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right)^2} \, dx
\]

\[= \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[= \int_{a}^{b} \sqrt{f'(t)^2 + g'(t)^2} \, dt.
\]

**THEOREM 10.8 ARC LENGTH IN PARAMETRIC FORM**

If a smooth curve \( C \) is given by \( x = f(t) \) and \( y = g(t) \) such that \( C \) does not intersect itself on the interval \( a \leq t \leq b \) (except possibly at the endpoints), then the arc length of \( C \) over the interval is given by

\[
s = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{a}^{b} \sqrt{f'(t)^2 + g'(t)^2} \, dt.
\]

**NOTE** When applying the arc length formula to a curve, be sure that the curve is traced out only once on the interval of integration. For instance, the circle given by \( x = \cos t \) and \( y = \sin t \) is traced out once on the interval \( 0 \leq t \leq 2\pi \), but is traced out twice on the interval \( 0 \leq t \leq 4\pi \).

In the preceding section you saw that if a circle rolls along a line, a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid**. The next example shows how to find the arc length of an epicycloid.
Try It 4

Find the arc length of the curve given by $x = t^2$ and $y = 4t^3 - 1$ on the interval $-1 \leq t \leq 1$.

Solution

Before applying Theorem 10.8, note in the figure that the curve has a sharp point when $t = 0$.

$$s = 2 \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Parametric form for arc length

$$= 2 \int_0^1 \sqrt{(2t)^2 + (12t^2)^2} \, dt$$

$$= 2 \int_0^1 \sqrt{4t^2 + 144t^4} \, dt$$

$$= 2 \int_0^1 2t \sqrt{1 + 36t^2} \, dt$$

$$= \frac{1}{18} \int_0^1 (1 + 36t^2)^{1/2} (72t) \, dt$$

$$= \left[ \frac{1}{27} (1 + 36t^2)^{3/2} \right]_0^1$$

$$\approx 8.30$$
EXAMPLE 5 Length of a Recording Tape

A recording tape 0.001 inch thick is wound around a reel whose inner radius is 0.5 inch and whose outer radius is 2 inches, as shown in Figure 10.34. How much tape is required to fill the reel?

Solution To create a model for this problem, assume that as the tape is wound around the reel, its distance \( r \) from the center increases linearly at a rate of 0.001 inch per revolution, or

\[
r = (0.001) \frac{\theta}{2\pi} = \frac{\theta}{2000\pi}, \quad 1000\pi \leq \theta \leq 4000\pi
\]

where \( \theta \) is measured in radians. You can determine that the coordinates of the point \((x, y)\) corresponding to a given radius are

\[
x = r \cos \theta
\]

and

\[
y = r \sin \theta.
\]

Substituting for \( r \), you obtain the parametric equations

\[
x = \left(\frac{\theta}{2000\pi}\right) \cos \theta \quad \text{and} \quad y = \left(\frac{\theta}{2000\pi}\right) \sin \theta.
\]

You can use the arc length formula to determine that the total length of the tape is

\[
s = \int_{1000\pi}^{4000\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]

\[
= \frac{1}{2000\pi} \int_{1000\pi}^{4000\pi} \sqrt{(- \theta \sin \theta + \cos \theta)^2 + (\theta \cos \theta + \sin \theta)^2} \, d\theta
\]

\[
= \frac{1}{2000\pi} \int_{1000\pi}^{4000\pi} \sqrt{\theta^2 + 1} \, d\theta
\]

\[
= \frac{1}{2000\pi} \left[ \frac{1}{2} \left( \theta \sqrt{\theta^2 + 1} + \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right) \right]_{1000\pi}^{4000\pi}
\]

\[
\approx 11,781 \text{ inches}
\]

\[
\approx 982 \text{ feet}.
\]

The length of the tape in Example 5 can be approximated by adding the circumferences of circular pieces of tape. The smallest circle has a radius of 0.501 and the largest has a radius of 2.

\[
s \approx 2\pi(0.501) + 2\pi(0.502) + 2\pi(0.503) + \cdots + 2\pi(2.000)
\]

\[
= \sum_{i=1}^{1500} 2\pi(0.5 + 0.001i)
\]

\[
= 2\pi \left[ 1500(0.5) + 0.001(1500)(1501)/2 \right]
\]

\[
\approx 11,786 \text{ inches}
\]
Try It 5

A recording tape 0.01 inch thick is wound around a reel whose inner radius is 0.5 inch and outer radius is 3 inches, as shown in the figure. How much tape is required to fill the reel?
Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

**THEOREM 10.9 AREA OF A SURFACE OF REVOLUTION**

If a smooth curve $C$ given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area $S$ of the surface of revolution formed by revolving $C$ about the coordinate axes is given by the following.

1. $S = 2\pi \int_{a}^{b} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$  \hspace{1cm} \text{Revolution about the x-axis: } g(t) \geq 0$
2. $S = 2\pi \int_{a}^{b} f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$  \hspace{1cm} \text{Revolution about the y-axis: } f(t) \geq 0$

These formulas are easy to remember if you think of the differential of arc length as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$  

Then the formulas are written as follows.

1. $S = 2\pi \int_{a}^{b} g(t) \, ds$  \hspace{1cm} 2. $S = 2\pi \int_{a}^{b} f(t) \, ds$
Try It 6

Find the area of the surface generated by revolving the curve given by
\[ x = t^3 \text{ and } y = t + 2, \quad 1 \leq t \leq 2, \]
about the y-axis.

Solution

On the interval \(1 \leq t \leq 2\), the curve is smooth and \(y\) is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

\[
S = 2\pi \int_{1}^{2} t^3 \sqrt{9t^4 + 1} \, dt
\]

\[
= \left[ \frac{\pi}{27} (9t^4 + 1)^{3/2} \right]_{1}^{2}
\]

\[
= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10})
\]

\[\approx 199.48\]
10.3 Exercises

In Exercises 1–4, find dy/dx.

1. \( x = t^2, \ y = 7 - 6t \)
2. \( x = \sqrt{t}, \ y = 4 - t \)
3. \( x = \sin^3 \theta, \ y = \cos^2 \theta \)
4. \( x = 2e^t, \ y = e^{-4t} \)

In Exercises 5–14, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. ( x = 4t, \ y = 3t - 2 )</td>
<td>( t = 3 )</td>
</tr>
<tr>
<td>6. ( x = \sqrt{t}, \ y = 3t - 1 )</td>
<td>( t = 1 )</td>
</tr>
<tr>
<td>7. ( x = t + 1, \ y = t^2 + 3t )</td>
<td>( t = -1 )</td>
</tr>
<tr>
<td>8. ( x = t^2 + 3t + 4, \ y = 4t )</td>
<td>( t = 0 )</td>
</tr>
<tr>
<td>9. ( x = 4 \cos \theta, \ y = 4 \sin \theta )</td>
<td>( \theta = \frac{\pi}{4} )</td>
</tr>
<tr>
<td>10. ( x = \cos \theta, \ y = 3 \sin \theta )</td>
<td>( \theta = 0 )</td>
</tr>
<tr>
<td>11. ( x = 2 + \sec \theta, \ y = 1 + 2 \tan \theta )</td>
<td>( \theta = \frac{\pi}{6} )</td>
</tr>
<tr>
<td>12. ( x = \sqrt{t}, \ y = \sqrt{t - 1} )</td>
<td>( t = 2 )</td>
</tr>
<tr>
<td>13. ( x = \cos^3 \theta, \ y = \sin^3 \theta )</td>
<td>( \theta = \frac{\pi}{4} )</td>
</tr>
<tr>
<td>14. ( x = \theta - \sin \theta, \ y = 1 - \cos \theta )</td>
<td>( \theta = \pi )</td>
</tr>
</tbody>
</table>

In Exercises 15–18, find an equation of the tangent line at each given point on the curve.

15. \( x = 2 \cos \theta \)
   \( y = 2 \sin^2 \theta \)
16. \( x = 2 - 3 \cos \theta \)
   \( y = 3 + 2 \sin \theta \)

In Exercises 19–22, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>19. ( x = 6t, \ y = t^2 + 4 )</td>
<td>( t = 1 )</td>
</tr>
<tr>
<td>20. ( x = t - 2, \ y = \frac{1}{t} + 3 )</td>
<td>( t = 1 )</td>
</tr>
</tbody>
</table>

In Exercises 23–26, find the equations of the tangent lines at the point where the curve crosses itself.

23. \( x = 2 \sin 2t, \ y = 3 \sin t \)
24. \( x = 2 - \pi \cos t, \ y = 2t - \pi \sin t \)
25. \( x = t^2 - t, \ y = t^3 - 3t - 1 \)
26. \( x = t^3 - 6t, \ y = t^3 \)

In Exercises 27 and 28, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.

27. Involute of a circle:
   \( x = \cos \theta + \theta \sin \theta \)
   \( y = \sin \theta - \theta \cos \theta \)
28. \( x = 2\theta \)
   \( y = 2(1 - \cos \theta) \)

In Exercises 29–38, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

29. \( x = 4 - t, \ y = t^3 \)
30. \( x = t + 1, \ y = t^2 + 3t \)
31. \( x = t + 4, \ y = t^3 - 3t \)
32. \( x = t^3 - t + 2, \ y = t^3 - 3t \)
33. \( x = 3 \cos \theta, \ y = 3 \sin \theta \)
34. \( x = \cos \theta, \ y = 2 \sin 2\theta \)
35. \( x = 5 + 3 \cos \theta, \ y = -2 + \sin \theta \)
36. \( x = 4 \cos^2 \theta, \ y = 2 \sin \theta \)
37. \( x = \sec \theta, \ y = \tan \theta \)
38. \( x = \cos^2 \theta, \ y = \cos \theta \)

In Exercises 39–44, determine the \( t \) intervals on which the curve is concave downward or concave upward.

39. \( x = 3t^2, \ y = t^3 - t \)
40. \( x = 2 + t^3, \ y = t^3 + t^3 \)
41. \( x = 2t + \ln t, \ y = 2t - \ln t \)
42. \( x = t^3, \ y = \ln t \)
43. \( x = \sin t, \ y = \cos t, \ 0 < t < \pi \)
44. \( x = 4 \cos t, \ y = 2 \sin t, \ 0 < t < 2\pi \)
Arc Length In Exercises 45–48, write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>45. ( x = 3t - t^2 ) ( y = 2t^{1/2} )</td>
<td>( 1 \leq t \leq 3 )</td>
</tr>
<tr>
<td>46. ( x = \ln t ) ( y = 4t - 3 )</td>
<td>( 1 \leq t \leq 5 )</td>
</tr>
<tr>
<td>47. ( x = e^{t^2} ) ( y = 2t + 1 )</td>
<td>(-2 \leq t \leq 2 )</td>
</tr>
<tr>
<td>48. ( x = t + \sin t ) ( y = t - \cos t )</td>
<td>( 0 \leq t \leq \pi )</td>
</tr>
</tbody>
</table>

Arc Length In Exercises 49–56, find the arc length of the curve on the given interval.

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>49. ( x = 3t^2 + 5 ) ( y = 7 - 2t )</td>
<td>(-1 \leq t \leq 3 )</td>
</tr>
<tr>
<td>50. ( x = t^2 ) ( y = 2t )</td>
<td>( 0 \leq t \leq 2 )</td>
</tr>
<tr>
<td>51. ( x = 6t^3 ) ( y = 2t^3 )</td>
<td>( 1 \leq t \leq 4 )</td>
</tr>
<tr>
<td>52. ( x = t^4 + 1 ) ( y = 4t^{3/2} + 3 )</td>
<td>(-1 \leq t \leq 0 )</td>
</tr>
<tr>
<td>53. ( x = e^{t^2} \cos t ) ( y = e^{-t} \sin t )</td>
<td>( 0 \leq t \leq \frac{\pi}{2} )</td>
</tr>
<tr>
<td>54. ( x = \arcsin t ) ( y = \ln \sqrt{1 - t^2} )</td>
<td>( 0 \leq t \leq \frac{1}{2} )</td>
</tr>
<tr>
<td>55. ( x = \sqrt{t} ) ( y = 3t - 1 )</td>
<td>( 0 \leq t \leq 1 )</td>
</tr>
<tr>
<td>56. ( x = t ) ( y = \frac{t^2}{10} + \frac{1}{6t^3} )</td>
<td>( 1 \leq t \leq 2 )</td>
</tr>
</tbody>
</table>

63. Folium of Descartes Consider the parametric equations:

\[
x = \frac{4t}{1 + t^2} \quad \text{and} \quad y = \frac{4t^2}{1 + t^2},
\]

(a) Use a graphing utility to graph the curve represented by the parametric equations.
(b) Use a graphing utility to find the points of horizontal tangency to the curve.
(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (Hint: Use symmetry and integrate over the interval \( 0 \leq t \leq 1 \).)

64. Witch of Agnesi Consider the parametric equations:

\[
x = 4 \cos \theta \quad \text{and} \quad y = 4 \sin^2 \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.
\]

(a) Use a graphing utility to graph the curve represented by the parametric equations.
(b) Use a graphing utility to find the points of horizontal tangency to the curve.
(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval \( \pi/4 \leq \theta \leq \pi/2 \).

65. Writing

(a) Use a graphing utility to graph each set of parametric equations.

\[
x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad 0 \leq t \leq \pi
\]

(b) Compare the graphs of the two sets of parametric equations in part (a). If the curve represents the motion of a particle and \( t \) is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?
(c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) if the path is modeled by

\[
x = \frac{t}{2} - \sin \left(\frac{t}{2}\right) \quad \text{and} \quad y = 1 - \cos \left(\frac{t}{2}\right).
\]

66. Writing

(a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

- **First Particle**
  
  \[
x = 3 \cos t \quad \text{and} \quad y = 4 \sin t
\]

- **Second Particle**
  
  \[
x = 4 \cos t \quad \text{and} \quad y = 3 \sin t
\]

(b) Determine the number of points of intersection.
(c) Will the particles ever be at the same place at the same time? If so, identify the point(s).
(d) Explain what happens if the motion of the second particle is represented by

\[
x = 2 + 3 \sin t \quad \text{and} \quad y = 2 - 4 \cos t, \quad 0 \leq t \leq 2\pi.
\]
Surface Area  In Exercises 67–70, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>67. ( x = 3t ), ( y = t^2 )</td>
<td>( 0 \leq t \leq 4 )</td>
</tr>
<tr>
<td>68. ( x = \frac{1}{4}t^2 ), ( y = t^2 )</td>
<td>( 0 \leq t \leq 3 )</td>
</tr>
<tr>
<td>69. ( x = \cos^2 \theta ), ( y = \cos \theta )</td>
<td>( 0 \leq \theta \leq \frac{\pi}{2} )</td>
</tr>
<tr>
<td>70. ( x = \theta + \sin \theta ), ( y = \theta + \cos \theta )</td>
<td>( 0 \leq \theta \leq \frac{\pi}{2} )</td>
</tr>
</tbody>
</table>

Surface Area  In Exercises 71–76, find the area of the surface generated by revolving the curve about each given axis.

| \( x = 2t \), \( y = 3t \) | \( 0 \leq t \leq 3 \) | (a) \( x \)-axis  (b) \( y \)-axis |
| \( x = t \), \( y = 4 - 2t \) | \( 0 \leq t \leq 2 \) | (a) \( x \)-axis  (b) \( y \)-axis |
| \( x = 5 \cos \theta \), \( y = 5 \sin \theta \) | \( 0 \leq \theta \leq \frac{\pi}{2} \) | (a) \( x \)-axis  (b) \( y \)-axis |
| \( x = \frac{3}{2}t \), \( y = t + 1 \) | \( 1 \leq t \leq 2 \) | \( y \)-axis |
| \( x = a \cos^3 \theta \), \( y = a \sin \theta \) | \( 0 \leq \theta \leq \pi \) | \( x \)-axis |
| \( x = a \cos \theta \), \( y = b \sin \theta \) | \( 0 \leq \theta \leq 2\pi \) | (a) \( x \)-axis  (b) \( y \)-axis |

WRITING ABOUT CONCEPTS

77. Give the parametric form of the derivative.

In Exercises 78 and 79, mentally determine \( \frac{dy}{dx} \).

78. \( x = t \), \( y = 3 \)  79. \( x = t \), \( y = 6t - 5 \)

80. Give the integral formula for arc length in parametric form.

81. Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve \( C \) is revolved about (a) the \( x \)-axis and (b) the \( y \)-axis.

CAPSTONE

82. (a) Sketch a graph of a curve defined by the parametric equations \( x = g(t) \) and \( y = f(t) \) such that \( dx/dt > 0 \) and \( dy/dt < 0 \) for all real numbers \( t \).

(b) Sketch a graph of a curve defined by the parametric equations \( x = g(t) \) and \( y = f(t) \) such that \( dx/dt < 0 \) and \( dy/dt > 0 \) for all real numbers \( t \).

83. Use integration by substitution to show that if \( y \) is a continuous function of \( x \) on the interval \( a \leq x \leq b \), where \( x = f(t) \) and \( y = g(t) \), then

\[
\int_a^b y \, dx = \int_{t_1}^{t_2} g(t)f'(t) \, dt
\]

where \( f(t_1) = a, f(t_2) = b \), and both \( g \) and \( f' \) are continuous on \([t_1, t_2]\).

84. Surface Area  A portion of a sphere of radius \( r \) is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of \( 2\theta \). Find the surface area removed from the sphere.

Area  In Exercises 85 and 86, find the area of the region. (Use the result of Exercise 83.)

85. \( x = 2 \sin^2 \theta \)  \( y = 2 \sin^2 \theta \tan \theta \) \( 0 \leq \theta < \frac{\pi}{2} \)

86. \( x = 2 \cot \theta \)  \( y = 2 \sin^2 \theta \) \( 0 < \theta < \pi \)

Areas of Simple Closed Curves  In Exercises 87–92, use a computer algebra system and the result of Exercise 83 to match the closed curve with its area. (These exercises were adapted from the article "The Surveyor's Area Formula" by Bart Braden in the September 1986 issue of the College Mathematics Journal, by permission of the author.)

(a) \( \frac{3}{4}ab \)  (b) \( \frac{1}{2}ma^2 \)  (c) \( 2\pi a^2 \)

(d) \( \pi ab \)  (e) \( \frac{e}{2}ab \)  (f) \( 6\pi a^2 \)

87. Ellipse: \( 0 \leq t \leq 2\pi \)  \( x = b \cos t \)  \( y = a \sin t \)

88. Astroid: \( 0 \leq t \leq 2\pi \)  \( x = a \cos^3 t \)  \( y = a \sin^3 t \)

89. Cardioid: \( 0 \leq t \leq 2\pi \)  \( x = 2a \cos t - a \cos 2t \)  \( y = 2a \sin t - a \sin 2t \)

90. Deltoid: \( 0 \leq t \leq 2\pi \)  \( x = 2a \cos t + a \cos 2t \)  \( y = 2a \sin t - a \sin 2t \)
91. Hourglass: \(0 \leq t \leq 2\pi\)
\[
x = a \sin 2t \\
y = b \sin t
\]
92. Teardrop: \(0 \leq t \leq 2\pi\)
\[
x = 2a \cos t - a \sin 2t \\
y = b \sin t
\]

**Centroid** In Exercises 93 and 94, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 83.)

93. \(x = \sqrt{t}, \ y = 4 - t\)
94. \(x = \sqrt{4 - t}, \ y = \sqrt{t}\)

**Volume** In Exercises 95 and 96, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the y-axis. (Use the result of Exercise 83.)

95. \(x = 6 \cos \theta, \ y = 6 \sin \theta\)
96. \(x = \cos \theta, \ y = 3 \sin \theta, \ a > 0\)

97. **Cycloid** Use the parametric equations
\[
x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta), a > 0
\]
to answer the following.
(a) Find \(dy/dx\) and \(d^2y/dx^2\).
(b) Find the equation of the tangent line at the point where \(\theta = \pi/6\).
(c) Find all points (if any) of horizontal tangency.
(d) Determine where the curve is concave upward or concave downward.
(e) Find the length of one arc of the curve.

98. Use the parametric equations
\[
x = t^2 \sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3
\]
to answer the following.
(a) Use a graphing utility to graph the curve on the interval \(-3 \leq t \leq 3\).
(b) Find \(dy/dx\) and \(d^2y/dx^2\).
(c) Find the equation of the tangent line at the point \((\sqrt{3}, \frac{4}{3})\).
(d) Find the length of the curve.
(e) Find the surface area generated by revolving the curve about the x-axis.

99. **Involute of a Circle** The involute of a circle is described by the endpoint \(P\) of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is
\[
x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).
\]

**Figure for 99**

**Figure for 100**

100. **Involute of a Circle** The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

(a) Use a graphing utility to graph the curve given by
\[
x = \frac{1 - t^2}{1 + t^2}, \ y = \frac{-2t}{1 + t^2}, \ -20 \leq t \leq 20.
\]
(b) Describe the graph and confirm your result analytically.
(c) Discuss the speed at which the curve is traced as \(t\) increases from \(-20\) to \(20\).

102. **Tractrix** A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point \((12, 0)\).

(a) In Exercise 96 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation
\[
y = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}
\]
where \(0 < x \leq 12\). Use a graphing utility to graph the rectangular equation.
(b) Use a graphing utility to graph the parametric equations
\[
x = 12 \text{sech } \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}
\]
where \(t \geq 0\). How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?
(c) Use the parametric equations for the tractrix to verify that the distance from the y-intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

**True or False?** In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

103. If \(x = f(t)\) and \(y = g(t)\), then \(d^2y/dx^2 = g''(t)/f'(t)\).
104. The curve given by \(x = t^2, y = t^3\) has a horizontal tangent at the origin because \(dy/dt = 0\) when \(t = 0\).
105. **Recording Tape** Another method you could use to solve Example 5 is to find the area of the reel with an inner radius of 0.5 inch and an outer radius of 2 inches, and then use the formula for the area of the rectangle where the width is 0.001 inch. Use this method to determine how much tape is required to fill the reel.