5.6 Inverse Trigonometric Functions: Differentiation

- Develop properties of the six inverse trigonometric functions.
- Differentiate an inverse trigonometric function.
- Review the basic differentiation rules for elementary functions.

Inverse Trigonometric Functions

This section begins with a rather surprising statement: None of the six basic trigonometric functions has an inverse function. This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the restricted domains.

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$ (see Figure 5.28). On this interval you can define the inverse of the restricted sine function as

$$y = \arcsin x \text{ if and only if } \sin y = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq \arcsin x \leq \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

### DEFINITIONS OF INVERSE TRIGONOMETRIC FUNCTIONS

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \arcsin x$ iff $\sin y = x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$-\pi/2 \leq y \leq \pi/2$</td>
</tr>
<tr>
<td>$y = \arccos x$ iff $\cos y = x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$0 \leq y \leq \pi$</td>
</tr>
<tr>
<td>$y = \arctan x$ iff $\tan y = x$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
<td>$-\pi/2 &lt; y &lt; \pi/2$</td>
</tr>
<tr>
<td>$y = \arccot x$ iff $\cot y = x$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
<td>$0 &lt; y &lt; \pi$</td>
</tr>
<tr>
<td>$y = \arcsec x$ iff $\sec y = x$</td>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$y = \arccsc x$ iff $\csc y = x$</td>
<td>$</td>
<td>x</td>
</tr>
</tbody>
</table>

**NOTE** The term "iff" is used to represent the phrase "if and only if."

**NOTE** The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x.$"
The graphs of the six inverse trigonometric functions are shown in Figure 5.29.

\[
\begin{align*}
\text{Domain: } & [-1, 1] \\
\text{Range: } & [-\pi/2, \pi/2] \\
\end{align*}
\]

\[
\begin{align*}
\text{Domain: } & [-1, 1] \\
\text{Range: } & [0, \pi] \\
\end{align*}
\]

\[
\begin{align*}
\text{Domain: } & (-\infty, \infty) \\
\text{Range: } & (-\pi/2, \pi/2) \\
\end{align*}
\]

\[
\begin{align*}
\text{Domain: } & (-\infty, -1] \cup [1, \infty) \\
\text{Range: } & [-\pi/2, 0) \cup (0, \pi/2] \\
\end{align*}
\]

\[
\begin{align*}
\text{Domain: } & (-\infty, -1] \cup [1, \infty) \\
\text{Range: } & [0, \pi/2) \cup (\pi/2, \pi] \\
\end{align*}
\]

\[
\begin{align*}
\text{Domain: } & (-\infty, \infty) \\
\text{Range: } & (0, \pi) \\
\end{align*}
\]

\[\text{arccsc} x = \arcsin \left( \frac{1}{x} \right)\]

\[\text{arcsec} x = \arccos \left( \frac{1}{x} \right)\]

\[\text{arc cot} x = \arctan \left( \frac{1}{x} \right) \quad x > 0\]

\[\pi + \arctan \left( \frac{1}{x} \right) \quad x < 0\]
Try It 1

Evaluate \( \arccot(-1) \).

\[
\begin{align*}
\arccot(-1) &= \arctan\left(\frac{1}{-1}\right) + \pi \\
&= -\frac{\pi}{4} + \pi \\
&= \frac{3\pi}{4}
\end{align*}
\]

\( x < 0 \)
Inverse functions have the properties
\[ f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x. \]

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For \( x \)-values outside these domains, these two properties do not hold. For example, \( \arcsin(\sin \pi) \) is equal to 0, not \( \pi \).

**Properties of Inverse Trigonometric Functions**

- If \(-1 \leq x \leq 1\) and \(-\pi/2 \leq y \leq \pi/2\), then
  \[ \sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y. \]
- If \(-\pi/2 < y < \pi/2\), then
  \[ \tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y. \]
- If \(|x| \geq 1\) and \(0 \leq y < \pi/2\) or \(\pi/2 < y \leq \pi\), then
  \[ \sec(\arcsec x) = x \quad \text{and} \quad \arcsec(\sec y) = y. \]

Similar properties hold for the other inverse trigonometric functions.

**Try It 2**

Solving an Equation

Solve \( \arctan 2x = -1 \) for \( x \).

\[
\begin{align*}
\arctan (2x) &= -1 \\
\tan (\arctan (2x)) &= \tan (-1) \\
2x &= \frac{\tan (-1)}{2} \\
x &= \frac{\tan (-1)}{2} = \frac{-0.1745}{2} = -0.0873 \\
x &= -0.0873
\end{align*}
\]
Using Right Triangles

**Try It 3**

Given $y = \arctan(3x)$, where $0 < y < \frac{\pi}{2}$, find $\sec y$.

\[ y = \arctan(3x) \]
\[ \tan y = \tan(\arctan(3x)) \]
\[ \tan y = \frac{3x}{1} \quad \text{tan } y = \frac{\text{opposite}}{\text{adjacent}} \]
\[ \sec(y) = \frac{1}{\cos y} = \frac{\sqrt{1+9x^2}}{1} \quad \sec y = \frac{\text{hyp}}{\text{adj}} \]
Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the transcendental function \( f(x) = \ln x \) is the algebraic function \( f'(x) = \frac{1}{x} \). You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Proofs for \( \arcsin u \) and \( \arccos u \) are given in Appendix A, and the rest are left as an exercise. (See Exercise 104.) Note that the derivatives of \( \arccos u \), \( \arccot u \), and \( \text{arccsc} \ u \) are the negatives of the derivatives of \( \arcsin u \), \( \text{arctan} \ u \), and \( \text{arcsec} \ u \), respectively.

**THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS**

Let \( u \) be a differentiable function of \( x \).

\[
\begin{align*}
\frac{d}{dx} [\arcsin u] &= \frac{u'}{\sqrt{1 - u^2}} \\
\frac{d}{dx} [\arccos u] &= \frac{-u'}{\sqrt{1 - u^2}} \\
\frac{d}{dx} [\arctan u] &= \frac{u'}{1 + u^2} \\
\frac{d}{dx} [\arccot u] &= \frac{-u'}{1 + u^2} \\
\frac{d}{dx} [\text{arcsec} \ u] &= \frac{u'}{|u|\sqrt{u^2 - 1}} \\
\frac{d}{dx} [\text{arccsc} \ u] &= \frac{-u'}{|u|\sqrt{u^2 - 1}}
\end{align*}
\]

**NOTE** There is no common agreement on the definition of \( \text{arccsc} \ x \) (or \( \text{arcsec} \ x \)) for negative values of \( x \). When we defined the range of the arccosecant, we chose to preserve the reciprocal identity

\[
\text{arcsec} \ x = \arccos \left( \frac{1}{x} \right).
\]

For example, to evaluate \( \text{arcsec} (-2) \), you can write

\[
\text{arcsec} (-2) = \arccos (-0.5) = 2.09.
\]

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every \( x \)-value in its domain. (See Figure 5.3.) This accounts for the absolute value sign in the formula for the derivative of \( \text{arcsec} \ x \).
Try It 4

Find the derivative of $f(x) = x \arctan x$.

\[
\frac{d}{dx} \left[ x \cdot \arctan x \right] = x \cdot \frac{d}{dx} [\arctan x] + \frac{d}{dx} [x] \cdot \arctan x
\]
\[
= x \cdot \left[ \frac{1}{1+x^2} \right] + \frac{1}{1} (\arctan x)
\]
\[
f'(x) = \frac{x}{1+x^2} + \arctan x
\]
Try It 5

Differentiate $y = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin x)$.

\[
y' = \frac{1}{2} \left[ x \cdot \frac{d}{dx} \left(1-x^2\right)^{\frac{3}{2}} + \frac{d}{dx}(1-x^2) \cdot (1-x^2)^{\frac{1}{2}} + \frac{d}{dx} \left[ \arcsin x \right] \right]
\]

\[
y' = \frac{1}{2} \left[ x \cdot \frac{1}{2}(1-x^2) (-2x) + 1 (1-x^2)^{\frac{1}{2}} + \frac{1}{\sqrt{1-x^2}} \right]
\]

\[
\frac{1}{2} \left[ \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} \right]
\]

Add 1st & 3rd Terms

\[
\frac{1}{2} \left[ \frac{1-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right]
\]

\[
\frac{(\sqrt{1-x^2})^2}{\sqrt{1-x^2}}
\]

\[
\frac{1}{2} \left[ \sqrt{1-x^2} + \sqrt{1-x^2} \right] = \frac{1}{2} \left[ 2\sqrt{1-x^2} \right] = \sqrt{1-x^2}
\]
Try It 6

Analyze the graph of \( y = \arccos(x) + 2 \arctan(x) \).

\[
y' = \frac{-1}{\sqrt{1-x^2}} + 2 \left( \frac{1}{1+x^2} \right)
\]

\[
y' = \frac{-1}{\sqrt{1-x^2}} + \frac{2}{1+x^2} = 0
\]

\[
\frac{-1(1+x^2) + 2\sqrt{1-x^2}}{\sqrt{1-x^2}(1+x^2)} = 0
\]

\[
-1(1+x^2) + 2\sqrt{1-x^2} = 0
\]

\[
2\sqrt{1-x^2} = 1 + x^2
\]

\[
\left( \frac{\sqrt{1-x^2}}{x} \right)^2 \left( \frac{1+x^2}{2} \right)^2
\]

\[
4(1-x^2) = 1 + 2x^2 + x^4,
\]

\[
4 - 9x^2 = x^4 + 2x^2 + 1 = 0
\]

\[
x^4 + 6x^2 - 3 = 0
\]

\[
u = x^2
\]

\[
u^2 + 6u - 3 = 0
\]

\[
u = \frac{-6 \pm \sqrt{36-4(1)(-3)}}{2(1)}
\]

\[
u = \frac{-6 \pm \sqrt{48}}{2}
\]

\[
u = -3 \pm 2\sqrt{3}
\]

\[
x^2 = \frac{-3 \pm 2\sqrt{3}}{2}
\]

\[
x = \pm \sqrt{-3 + 2\sqrt{3}}
\]

\[
x = \pm 1.681
\]

First Derivative Test
Try It 7

A photographer is taking a picture of a two-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting, as shown in the figure. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

\[ \beta \text{ is to be maximized!} \]

\[ \beta = \Theta - \alpha \]

\[ = \arccot \frac{x}{3} - \arccot x \]

Differentiate

\[ \frac{d\beta}{dx} = -\frac{\frac{1}{2}}{1 + \frac{x^2}{9}} - \frac{-1}{1 + x^2} \]

\[ = -\frac{3}{9 + x^2} + \frac{1}{1 + x^2} \]

\[ = \frac{2(3-x^2)}{(9+x^2)(1+x^2)} = 0 \]

\[ 3-x^2 = 0 \]

\[ x^2 = 3 \]

\[ x = \pm \sqrt{3} \]

\[ x = \sqrt{3} \]

Max: Min

\[ x = \sqrt{3} \quad \text{will maximize the angle } \beta \]
Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An elementary function is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

<table>
<thead>
<tr>
<th>Algebraic Functions</th>
<th>Transcendental Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial functions</td>
<td>Logarithmic functions</td>
</tr>
<tr>
<td>Rational functions</td>
<td>Exponential functions</td>
</tr>
<tr>
<td>Functions involving radicals</td>
<td>Trigonometric functions</td>
</tr>
<tr>
<td></td>
<td>Inverse trigonometric functions</td>
</tr>
</tbody>
</table>

With the differentiation rules introduced so far in the text, you can differentiate any elementary function. For convenience, these differentiation rules are summarized below.
**BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS**

1. \( \frac{d}{dx}[cu] = cu' \)

2. \( \frac{d}{dx}[u \pm v] = u' \pm v' \)

3. \( \frac{d}{dx}[uv] = uv' + vu' \)

4. \( \frac{d}{dx}\left[ \frac{u}{v} \right] = \frac{vu' - uv'}{v^2} \)

5. \( \frac{d}{dx}[c] = 0 \)

6. \( \frac{d}{dx}[u^n] = nu^{n-1}u' \)

7. \( \frac{d}{dx}[x] = 1 \)

8. \( \frac{d}{dx}[|u|] = \frac{u}{|u|}u', \quad u \neq 0 \)

9. \( \frac{d}{dx}[\ln u] = \frac{u'}{u} \)

10. \( \frac{d}{dx}[e^u] = e^u u' \)

11. \( \frac{d}{dx}[\log_a u] = \frac{u'}{\ln a u} \)

12. \( \frac{d}{dx}[a^u] = (\ln a)a^u u' \)

13. \( \frac{d}{dx}[\sin u] = (\cos u)u' \)

14. \( \frac{d}{dx}[\cos u] = -(\sin u)u' \)

15. \( \frac{d}{dx}[\tan u] = (\sec^2 u)u' \)

16. \( \frac{d}{dx}[\cot u] = -(\csc^2 u)u' \)

17. \( \frac{d}{dx}[\sec u] = (\sec u \tan u)u' \)

18. \( \frac{d}{dx}[\csc u] = -(\csc u \cot u)u' \)

19. \( \frac{d}{dx}[\arcsin u] = \frac{-u'}{\sqrt{1 - u^2}} \)

20. \( \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}} \)

21. \( \frac{d}{dx}[\arctan u] = \frac{-u'}{1 + u^2} \)

22. \( \frac{d}{dx}[\arccot u] = \frac{-u'}{1 + u^2} \)

23. \( \frac{d}{dx}[\arcsec u] = \frac{u'}{|u|\sqrt{u^2 - 1}} \)

24. \( \frac{d}{dx}[\arccsc u] = \frac{-u'}{|u|\sqrt{u^2 - 1}} \)
5.6 Exercises


Numerical and Graphical Analysis In Exercises 1 and 2, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>-0.8</th>
<th>-0.6</th>
<th>-0.4</th>
<th>-0.2</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. \( y = \arcsin x \)  
2. \( y = \arccos x \)

In Exercises 3 and 4, determine the missing coordinates of the points on the graph of the function.

3. \( y = \arcsin x \)  
4. \( y = \arccos x \)

In Exercises 5–12, evaluate the expression without using a calculator.

5. \( \arcsin \frac{1}{2} \)  
6. \( \arcsin 0 \)
7. \( \arccos \frac{1}{2} \)  
8. \( \arccos 1 \)
9. \( \arctan \frac{\sqrt{3}}{3} \)  
10. \( \arccos(-\sqrt{3}) \)
11. \( \arccsc(-\sqrt{2}) \)  
12. \( \arccsc(-\sqrt{2}) \)

In Exercises 13–16, use a calculator to approximate the value. Round your answer to two decimal places.

13. \( \arccos(-0.8) \)
14. \( \arcsin(-0.39) \)
15. \( \arccos 1.269 \)
16. \( \arctan(-5) \)

In Exercises 17–20, evaluate each expression without using a calculator. \( \text{[Hint: See Example 3.]} \)

17. \( \sin \left( \arctan \frac{3}{4} \right) \)
18. \( \tan \left( \arccos \frac{\sqrt{2}}{2} \right) \)
(b) \( \sec \left( \arcsin \frac{4}{5} \right) \)
19. \( \cot \left( \arccos \left( -\frac{1}{2} \right) \right) \)
(b) \( \csc \left( \arctan \frac{5}{12} \right) \)
20. \( \sec \left( \arctan \left( -\frac{3}{5} \right) \right) \)
(b) \( \tan \left( \arcsin \frac{5}{6} \right) \)

In Exercises 21–26, use the figure to write the expression in algebraic form given \( y = \arccos x \), where \( 0 < y < \pi/2 \).

21. \( \cos y \)
22. \( \sin y \)
23. \( \tan y \)
24. \( \cot y \)
25. \( \sec y \)
26. \( \csc y \)

In Exercises 27–34, write the expression in algebraic form. \( \text{[Hint: Sketch a right triangle, as demonstrated in Example 3.]} \)

27. \( \cos(\arcsin 2x) \)
28. \( \sec(\arctan 4x) \)
29. \( \sin(\arccsc x) \)
30. \( \cos(\arccot x) \)
31. \( \tan(\arcsin x) \)
32. \( \sec(\arcsin(x - 1)) \)
33. \( \csc(\arctan \frac{x}{\sqrt{2}}) \)
34. \( \cos(\arccos x + h) \)

In Exercises 35 and 36, (a) use a graphing utility to graph \( f \) and \( g \) in the same viewing window to verify that they are equal, (b) use algebra to verify that \( f \) and \( g \) are equal, and (c) identify any horizontal asymptotes of the graphs.

35. \( f(x) = \sin(\arctan x) \), \( g(x) = \frac{2x}{\sqrt{1 + 4x^2}} \)
36. \( f(x) = \tan(\arccos \frac{x}{2}) \), \( g(x) = \frac{\sqrt{4 - x^2}}{x} \)

In Exercises 37–40, solve the equation for \( x \).

37. \( \arcsin(3x - x^2) = \frac{\pi}{2} \)
38. \( \arctan(2x - 5) = -1 \)
39. \( \arcsin \frac{\pi}{2} = \arccos \frac{\pi}{2} \)
40. \( \arccos x + \arccos x = -1 \)

In Exercises 41 and 42, verify each identity.

41. \( a) \ \arccsc x = \arcsin \frac{1}{x} \), \( x \geq 1 \)
(b) \( \arctan x + \arctan \frac{1}{x} = \arctan 1 \), \( x > 0 \)
42. \( a) \ \arcsin(-x) = -\arcsin x \), \( |x| \leq 1 \)
(b) \( \arccos(-x) = \pi - \arccos x \), \( |x| \leq 1 \)

In Exercises 43–62, find the derivative of the function.

43. \( f(x) = 2 \arcsin(x - 1) \)
44. \( f(t) = \arcsin t^2 \)
45. \( g(x) = 3 \arccos \frac{x}{2} \)
46. \( f(x) = \arccos 2x \)
47. \( f(x) = \arctan e^x \)
48. \( f(x) = \arctan \sqrt{x} \)
49. \( g(x) = \arcsin \frac{3x}{x} \)
50. \( h(x) = x^2 \arctan 5x \)
51. \( h(t) = \sin(\arccos t) \)
52. \( f(x) = \arcsin x + \arccos x \)
53. \( y = 2x \arccos x - 2 \sqrt{1 - x^2} \)
54. \( y = \ln(x^2 + 4) - \frac{1}{2} \arctan \frac{1}{2} \)
55. \( y = \frac{1}{2} \left[ \ln \frac{x + 1}{x - 1} + \arctan x \right] \)
56. \( y = \frac{1}{2} \left[ \sqrt{4 - x^2} + 4 \arcsin \left( \frac{x}{2} \right) \right] \)
57. \( y = x \arccos x + \sqrt{1 - x^2} \)
58. \( y = x \arctan 2x - \frac{1}{4} \ln(1 + 4x^2) \)
59. \( y = 8 \arcsin \frac{x}{4} - \frac{x \sqrt{16 - x^2}}{2} \)
60. \( y = 25 \arcsin \frac{x}{5} - x \sqrt{25 - x^2} \)
61. \( y = \arctan x + \frac{x}{1 + x^2} \)
62. \( y = \arctan \frac{x}{2} - \frac{1}{2(x^2 + 4)} \)

In Exercises 63–68, find an equation of the tangent line to the graph of the function at the given point.

63. \( y = 2 \arcsin x, \quad \left( \frac{\pi}{3}, \frac{\sqrt{3}}{2} \right) \)
64. \( y = \frac{1}{2} \arccos x, \quad \left( \frac{\sqrt{2}}{2}, \frac{3\pi}{8} \right) \)
65. \( y = \arctan \frac{x}{2}, \quad \left( \frac{\pi}{4}, \frac{\sqrt{2}}{2} \right) \)
66. \( y = \arccsc 4x, \quad \left( \frac{\pi}{4}, \frac{\sqrt{2}}{4} \right) \)
67. \( y = 4x \arccos(x - 1), \quad (1, 2\pi) \)
68. \( y = 3x \arcsin x, \quad \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \)

**E45 Linear and Quadratic Approximations**

In Exercises 69–72, use a computer algebra system to find the linear approximation \( P_1(x) = f(a) + f'(a)(x - a) \) and the quadratic approximation \( P_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \) of the function \( f \) at \( x = a \). Sketch the graph of the function and its linear and quadratic approximations.

69. \( f(x) = \arctan x, \quad a = 0 \)
70. \( f(x) = \arccos x, \quad a = 0 \)
71. \( f(x) = \arcsin x, \quad a = \frac{1}{2} \)
72. \( f(x) = \arctan x, \quad a = 1 \)

In Exercises 73–76, find any relative extrema of the function.

73. \( f(x) = \arccsc x - x \)
74. \( f(x) = \arcsin x - 2x \)
75. \( f(x) = \arctan x - \arctan(x - 4) \)
76. \( h(x) = \arcsin x - 2 \arccos x \)

In Exercises 77–80, analyze and sketch a graph of the function. Identify any relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

77. \( f(x) = \arcsin(x - 1) \)
78. \( f(x) = \arctan x + \frac{\pi}{2} \)
79. \( f(x) = \arccsc 2x \)
80. \( f(x) = \arccos \frac{x}{4} \)

**Implicit Differentiation**

In Exercises 81–84, find an equation of the tangent line to the graph of the function at the given point.

81. \( x^2 + x \arctan y = y - 1, \quad \left( \frac{-\pi}{4}, 1 \right) \)
82. \( \arctan(y) = \arcsin(x + y), \quad (0, 0) \)
83. \( \arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2} \right) \)
84. \( \arctan(x + y) = x^2 + \frac{\pi}{4}, \quad (1, 0) \)

**WRITING ABOUT CONCEPTS**

85. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.
86. Explain why \( \tan z = 0 \) does not imply that \( \arctan 0 = z \).
87. Explain how to graph \( y = \arccos x \) on a graphing utility that does not have the arccotangent function.
88. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

89. (a) Use a graphing utility to evaluate \( \arcsin(\arcsin 0.5) \) and \( \arcsin(\arcsin 1) \).
   (b) Let \( f(x) = \arcsin(\arcsin x) \). Find the values of \( x \) in the interval \( -1 \leq x \leq 1 \) such that \( f(x) \) is a real number.

**CAPSTONE**

90. The point \( (\frac{3\pi}{2}, 0) \) is on the graph of \( y = \cos x \). Does \( \left( \frac{3\pi}{2}, 0 \right) \) lie on the graph of \( y = \arccos x \)? If not, does this contradict the definition of inverse function?

**True or False?**

In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

91. Because cos \( \left( \frac{\pi}{3} \right) = \frac{1}{2} \), it follows that \( \arccos \frac{1}{2} = \frac{\pi}{3} \)
92. \( \arccos \frac{\pi}{3} \neq \frac{\pi}{2} \)
93. The slope of the graph of the inverse tangent function is positive for all \( x \).
94. The range of \( y = \arcsin x \) is \([0, \pi]\).
95. \( \frac{d}{dx} [\arctan(\tan x)] = 1 \) for all \( x \) in the domain.
96. \( \arcsin^2 x + \arccos^2 x = 1 \)

**Angular Rate of Change**

An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider \( \theta \) and \( x \) as shown in the figure on the next page.
Figure for 97

(a) Write $\theta$ as a function of $x$.
(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.

98. Writing Repeat Exercise 97 for an altitude of 3 miles and describe how the altitude affects the rate of change of $\theta$.

99. Angular Rate of Change In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).
(a) Find the position function that yields the height of the object at time $t$ assuming the object is released at time $t = 0$. At what time will the object reach ground level?
(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

100. Angular Rate of Change A television camera at ground level is filming the lift-off of a space shuttle at a point 800 meters from the launch pad. Let $\theta$ be the angle of elevation of the shuttle and let $x$ be the distance between the camera and the shuttle (see figure). Write $\theta$ as a function of $x$ for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of $x$ and $dx/dt$.

101. Maximizing an Angle A billboard 85 feet wide is perpendicular to a straight road and is 40 feet from the road (see figure). Find the point on the road at which the angle $\theta$ subtended by the billboard is a maximum.

102. Angular Speed A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. Write $\theta$ as a function of $x$. How fast is the light beam moving along the wall when the beam makes an angle of $\theta = 45^\circ$ with the line perpendicular from the light to the wall?

(a) Prove that $\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$ for $xy \neq 1$.
(b) Use the formula in part (a) to show that
   \[
   \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}.
   \]

104. Verify each differentiation formula.
(a) $\frac{d}{dx} (\arctan u) = \frac{u'}{1 + u^2}$
(b) $\frac{d}{dx} (\text{arccot} u) = -\frac{u'}{1 + u^2}$
(c) $\frac{d}{dx} (\text{arcsec} u) = \frac{u'}{|u| \sqrt{u^2 - 1}}$
(d) $\frac{d}{dx} (\text{arcsec} u) = -\frac{u'}{|u| \sqrt{u^2 - 1}}$

106. Think About It Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin (\sin x)$.
(a) Why isn’t the graph of $g$ the line $y = x$?
(b) Determine the extrema of $g$.

107. (a) Graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$.
(b) Describe the graph of $f$.
(c) Verify the result of part (b) analytically.

108. Prove that $\arcsin x = \arctan \left( \frac{x}{\sqrt{1 - x^2}} \right)$ for $|x| < 1$.

109. In the figure find the value of $c$ in the interval $[0, 4]$ on the $x$-axis that maximizes angle $\theta$.

110. In the figure find $PR$ such that $0 \leq PR \leq 3$ and $m \perp \theta$ is a maximum.

111. Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup (\pi, 3\pi/2)$.
(a) Sketch the graph $y = \text{arcsec} x$ using this range.
(b) Show that $y' = \frac{1}{x \sqrt{x^2 - 1}}$. 

Figure for 109

Figure for 110

Figure for 102