Section 5.8

\[ y = \sinh^{-1}(\tan x) \]

\[ u = \tan x \quad \frac{du}{dx} = \sec^2 x \]

\[ y' = \frac{\sec^2 x}{\sqrt{\tan^2 x + 1}} \]

\[ y' = \sqrt{\sec^2 x} = \sec x \]

\[ |\sec x| \]
\[ y = \cosh(x) \]

\[ y = \cosh^{-1}(x) \]

\[ \cosh y = \frac{e^y + e^{-y}}{2} \]

\[ x = \cosh(y) = \frac{e^y + e^{-y}}{2} \]

\[ 2x = e^y + e^{-y} \]

\[ e^y \{ e^y - 2x + e^{-y} = 0 \} \]

Multiply by \( e^y \)

\[ e^{2y} - 2xe^y + 1 = 0 \]

\[ (e^y)^2 - 2x(e^y) + 1 = 0 \]

\[ q = 1 \quad b = -2x \quad c = 1 \]

\[ e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} \]

\[ e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2x \pm 2\sqrt{x^2 - 4}}{2} \]

\[ e^y = x \pm \sqrt{x^2 - 1} \]

\[ \ln e^y = \ln(x \pm \sqrt{x^2 - 1}) \]

\[ y = \ln(x \pm \sqrt{x^2 - 1}) \]

\[ \frac{d}{dx} \left[ \cosh(x) \right] = \frac{1}{x \pm \sqrt{x^2 - 1}} \]

\[ = \frac{1}{\sqrt{x^2 - 1}} \]
Derivative of $\sin^{-1} x$

$y = \sin^{-1}(x)$

$\sin y = \sin \sin^{-1}(x)$

$\sin y = \frac{x}{\sqrt{1-x^2}}$

$\cos y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
\[ A = 2 \int_{0}^{1} \sec h \left( \frac{x}{2} \right) \, dx \]

Using the identity \( \sec h(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \),

\[ = 2 \int_{0}^{1} \frac{\frac{2 \, dx}{e^x + e^{-x}}}{e^x - e^{-x}} \]

Substituting \( u = e^{x/2} \), \( du = \frac{1}{2} e^{x/2} \, dx \),

\[ = 8 \int_{0}^{1} \frac{1}{2} e^{x/2} \, dx \]

Using the identity \( \frac{1}{2} \arctan(u) \),

\[ = 8 \left[ \frac{1}{2} \arctan \left( \frac{e^{x/2}}{1} \right) \right]_{0}^{1} \]

\[ = 8 \left[ \frac{1}{2} \arctan(e^{1/2}) - \frac{1}{2} \arctan(1) \right] \]

\[ = 8 \left[ \frac{1}{2} \arctan(e^{1/2}) - \frac{1}{2} \pi \right] \]

\[ = 8 \arctan(e^{1/2}) - 2\pi \]
General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. Recall that a differential equation in $x$ and $y$ is an equation that involves $x$, $y$, and derivatives of $y$. In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a solution of a differential equation if the equation is satisfied when $y$ and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form $y = C e^{-2x}$, where $C$ is any real number. This solution is called the general solution. Some differential equations have singular solutions that cannot be written as special cases of the general solution. However, such solutions are not considered in this text. The order of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation. First-order linear differential equations are discussed in Section 6.4.

In Section 4.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution $s(t) = -16t^2 + C_1 t + C_2$, which contains two arbitrary constants. It can be shown that a differential equation of order $n$ has a general solution with $n$ arbitrary constants.
Try It 1

Determine whether the function, \( y = 5e^{-2x} + 3 \cos 2x \), is a solution of the differential equation \( y^{(4)} - 16y = 0 \).

\[
\begin{align*}
y &= 5e^{-2x} + 3 \cos 2x \\
y' &= -10e^{-2x} - 6 \sin (2x) \\
y'' &= 20e^{-2x} - 12 \cos (2x) \\
y''' &= -40e^{-2x} + 24 \sin (2x) \\
y^{(4)} &= 80e^{-2x} - 48 \cos (2x)
\end{align*}
\]

\[y^{(4)} - 16y = 0\] \(\rightarrow\) 4th Order

\[
\begin{align*}
[80e^{-2x} + 48 \cos (2x)] - 16 \left( 5e^{-2x} + 3 \cos (2x) \right) \\
80e^{-2x} + 48 \cos (2x) - 80e^{-2x} - 48 \cos (2x) = 0
\end{align*}
\]
Geometrically, the general solution of a first-order differential equation represents a family of curves known as solution curves, one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form
\[ y = \frac{C}{x} \]
is a solution of the differential equation \( xy' + y = 0 \). Figure 6.1 shows four of the solution curves corresponding to different values of \( C \).

As discussed in Section 4.1, particular solutions of a differential equation are obtained from initial conditions that give the values of the dependent variable or one of its derivatives for particular values of the independent variable. The term “initial condition” stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the initial time \( t = 0 \). For instance, the second-order differential equation \( s''(t) = -32 \) having the general solution
\[ s(t) = -16t^2 + C_1t + C_2 \]
might have the following initial conditions.
\[ s(0) = 80, \quad s'(0) = 64 \]
In this case, the initial conditions yield the particular solution
\[ s(t) = -16t^2 + 64t + 80. \]
Try It 2

For the differential equation \( 2x + 3yy' = 0 \)

verify that \( 2x^2 + 3y^2 = C \) is a solution, and find the particular solution determined by the initial condition \( y = 2 \) when \( x = 1 \).

**Check**

\[
2x + 3y y' = 0 \\
2x + 3y \left( \frac{-2x}{3y} \right) = 0 \\
2x - 2x = 0
\]

Checks \( \checkmark \)

**Initial Condition**

\( 2x^2 + 3y^2 = C \)

\[
2(1)^2 + 3(2)^2 = C \\
2 + 12 = C \\
14 = C
\]

**Equation of Ellipse**

\( 2x^2 + 3y^2 = 14 \)
Slope Fields

Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

\[ y' = F(x, y) \]

Differential equation

where \( F(x, y) \) is some expression in \( x \) and \( y \). At each point \( (x, y) \) in the \( xy \)-plane where \( F \) is defined, the differential equation determines the slope \( y' = F(x, y) \) of the solution at that point. If you draw short line segments with slope \( F(x, y) \) at selected points \( (x, y) \) in the domain of \( F \), then these line segments form a slope field, or a direction field, for the differential equation \( y' = F(x, y) \). Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

Try It 3

Sketch a slope field for the differential equation \( y' = 2x - y^2 \) or the points \((-1, -1), (0, -1), \) and \((1, -1)\).
Try It 4

Match each slope field with its differential equation.

**a.** $y' = 2y$
- $y = 0$
- $y' = 0$

**b.** $y' = 2x - 2y$
- $x = y$

**c.** $y' = 2x$
- $x = 0$
- $y' = 0$
Try It 5

Sketch a slope field for the differential equation
\[ y' = 2 - y. \]
Use the slope field to sketch the solution that passes through the point (0, 4).

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-2</th>
<th>-1</th>
<th>-1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-3</td>
<td>3</td>
<td>-3</td>
<td>3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>( y' = 2 - y )</td>
<td>5</td>
<td>-1</td>
<td>5</td>
<td>-1</td>
<td>5</td>
<td>-1</td>
<td>5</td>
<td>-1</td>
<td>5</td>
<td>-1</td>
</tr>
</tbody>
</table>

Figure (a)

Figure (b)
Euler's Method

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

\[ y' = F(x, y) \]

that passes through the point \((x_0, y_0)\). From the given information, you know that the graph of the solution passes through the point \((x_0, y_0)\) and has a slope of \(F(x_0, y_0)\) at this point. This gives you a "starting point" for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step \(h\), move along the tangent line until you arrive at the point \((x_1, y_1)\), where

\[ x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0) \]

as shown in Figure 6.6. If you think of \((x_1, y_1)\) as a new starting point, you can repeat the process to obtain a second point \((x_2, y_2)\). The values of \(x_i\) and \(y_i\) are as follows.

\[
\begin{align*}
  x_1 &= x_0 + h \\
  y_1 &= y_0 + hF(x_0, y_0) \\
  x_2 &= x_1 + h \\
  y_2 &= y_1 + hF(x_1, y_1) \\
  & \vdots \\
  x_n &= x_{n-1} + h \\
  y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1})
\end{align*}
\]

**Note:** You can obtain better approximations of the exact solution by choosing smaller and smaller step sizes.
Try It 6

Use Euler’s Method to approximate the particular solution of the differential equation
\[ y' = x + y \]
passing through (0, 2). Use a step of \( h = 0.05 \).

Solution
Using \( h = 0.05 \), \( x_0 = 0, y_0 = 2 \), and \( F(x, y) = x + y \), you have \( x_0 = 0, x_1 = 0.05, x_2 = 0.1, x_3 = 0.15, \ldots \), and
\[
\begin{align*}
y_1 &= y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1 \\
y_2 &= y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075 \\
y_3 &= y_2 + hF(x_2, y_2) = 2.2075 + (0.05)(0.1 + 2.2075) = 2.3229
\end{align*}
\]
The first ten approximations are shown in the table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>( y_n )</td>
<td>2</td>
<td>2.208</td>
<td>2.447</td>
<td>2.720</td>
<td>3.032</td>
<td>3.387</td>
<td>3.788</td>
<td>4.240</td>
<td>4.749</td>
<td>5.320</td>
<td>5.960</td>
</tr>
</tbody>
</table>
### 6.1 Exercises


In Exercises 1–8, verify the solution of the differential equation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $y = Ce^{2x}$</td>
<td>$y' = 4y$</td>
</tr>
<tr>
<td>2. $y = e^{-2x}$</td>
<td>$3y' + 5y = -e^{-2x}$</td>
</tr>
<tr>
<td>3. $x^2 + y^2 = Cy$</td>
<td>$y' = \frac{2xy}{(x^2 - y^2)}$</td>
</tr>
<tr>
<td>4. $y^2 - 2\ln y = x^3$</td>
<td>$\frac{dy}{dx} = \frac{xy}{y^2 - 1}$</td>
</tr>
<tr>
<td>5. $y = C\sin x - C_2\cos x$</td>
<td>$y' + y = 0$</td>
</tr>
<tr>
<td>6. $y = C_1e^{-x} + C_2e^{-x}\sin x$</td>
<td>$y'' + 2y' + 2y = 0$</td>
</tr>
<tr>
<td>7. $y = -\cos x \ln(\sec x + \tan x)$</td>
<td>$y'' + y = \tan x$</td>
</tr>
<tr>
<td>8. $y = \frac{2}{3}(e^{-4x} + e^{3x})$</td>
<td>$y'' + 4y' = 2e^x$</td>
</tr>
</tbody>
</table>

In Exercises 9–12, verify the particular solution of the differential equation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation and Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. $y = \sin x \cos x - \cos^2 x$</td>
<td>$2y + y' = 2\sin(2x) - 1$</td>
</tr>
<tr>
<td></td>
<td>$\left(\frac{\pi}{4}\right)$</td>
</tr>
<tr>
<td>10. $y = \frac{1}{4}x^2 - 2\cos x - 3$</td>
<td>$y' = x + 2\sin x$</td>
</tr>
<tr>
<td></td>
<td>$y(0) = -5$</td>
</tr>
<tr>
<td>11. $y = 4e^{-3x}$</td>
<td>$y' = -12y$</td>
</tr>
<tr>
<td></td>
<td>$y(0) = 4$</td>
</tr>
<tr>
<td>12. $y = e^{-\cos x}$</td>
<td>$y' = y\sin x$</td>
</tr>
<tr>
<td></td>
<td>$\left(\frac{x}{2}\right)$</td>
</tr>
</tbody>
</table>

In Exercises 13–20, determine whether the function is a solution of the differential equation $y'' - 10y = 0$.

13. $y = 3\cos x$
14. $y = 2\sin x$
15. $y = 3\cos 2x$
16. $y = 3\sin 2x$
17. $y = e^{-2x}$
18. $y = 5\ln x$
19. $y = C_1e^{3x} + C_2e^{-3x} + C_3\sin 2x + C_4\cos 2x$
20. $y = 3e^{2x} - 4\sin 2x$

In Exercises 21–28, determine whether the function is a solution of the differential equation $xy' - 2y = x^2e^x$.

21. $y = x^2$
22. $y = x^3$
23. $y = xe^{2x}$
24. $y = x^2(2 + e^x)$
25. $y = \sin x$
26. $y = \cos x$
27. $y = \ln x$
28. $y = xe^{2x} - 5x^2$

In Exercises 29–32, some of the curves corresponding to different values of $C$ in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>29. $y = Ce^{-\frac{y}{x}}$</td>
<td>$2y' + y = 0$</td>
</tr>
<tr>
<td>30. $y(x^2 + y) = C$</td>
<td>$2xy + (x^2 + 2y)y' = 0$</td>
</tr>
<tr>
<td>31. $y^2 = Cx^5$</td>
<td>$2xy' - 3y = 0$</td>
</tr>
<tr>
<td>32. $2x^2 - y^3 = C$</td>
<td>$yy' - 2x = 0$</td>
</tr>
</tbody>
</table>

In Exercises 33 and 34, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of $C$.

33. $4yxy' - x = 0$
34. $y'y + x = 0$

- $4y^2 - x^2 = C$
- $x^2 + y^2 = C$
- $C = 0$, $C = \pm 1$, $C = \pm 4$

In Exercises 35–40, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

35. $y = Ce^{-2x}$
36. $3x^2 + 2y^2 = C$

- $y' + 2y = 0$
- $y = 3$ when $x = 0$
- $3x + 2yy' = 0$
- $y = 3$ when $x = 1$

37. $y = C_1\sin 3x + C_2\cos 3x$
38. $y = C_1 + C_2\ln x$

- $y'' + 9y = 0$
- $y = 0$ when $x = 2$
- $xy'' + y' = 0$
- $y = 0$ when $x = 2$
- $y'' = 1$ when $x = \pi/6$
- $y'' = 1$ when $x = 2$
- $y' = \frac{1}{2}$ when $x = 2$
39. \( y = C_1 x + C_2 x^3 \)  
   \( x^2 y'' - 3x y' + 3y = 0 \)  
   \( y = 0 \) when \( x = 2 \)  
   \( y' = 4 \) when \( x = 2 \)  
   \( y = 0 \) when \( x = 3 \)

In Exercises 41–52, use integration to find a general solution of the differential equation.

41. \( \frac{dy}{dx} = 6x^2 \)  
42. \( \frac{dy}{dx} = 2x^3 - 3x \)

43. \( \frac{dy}{dx} = \frac{x}{1 + x^3} \)  
44. \( \frac{dy}{dx} = \frac{e^y}{4 + e^y} \)

45. \( \frac{dy}{dx} = \frac{x - 2}{x} \)  
46. \( \frac{dy}{dx} = x \cos x^2 \)

47. \( \frac{dy}{dx} = \sin 2x \)  
48. \( \frac{dy}{dx} = \tan^2 x \)

49. \( \frac{dy}{dx} = x \sqrt{x - 6} \)  
50. \( \frac{dy}{dx} = 2x \sqrt{3 - x} \)

51. \( \frac{dy}{dx} = xe^{x^2} \)  
52. \( \frac{dy}{dx} = 5e^{-x^2} \)

Slope Fields In Exercises 53–56, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

53. \( \frac{dy}{dx} = \frac{2y}{x} \)  
54. \( \frac{dy}{dx} = y - x \)

55. \( \frac{dy}{dx} = x \cos \frac{dy}{8} \)  
56. \( \frac{dy}{dx} = \tan \left( \frac{dy}{6} \right) \)

In Exercises 57–60, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]

57. \( \frac{dy}{dx} = \sin(2x) \)  
58. \( \frac{dy}{dx} = \frac{1}{2} \cos x \)

59. \( \frac{dy}{dx} = e^{-3x} \)  
60. \( \frac{dy}{dx} = \frac{1}{x} \)

Slope Fields In Exercises 61–64, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as \( x \to \infty \) and \( x \to -\infty \). Use a graphing utility to verify your results.

61. \( y' = 3 - x \), (4, 2)  
62. \( y' = \frac{1}{2} x^2 - \frac{1}{3} x \), (1, 1)  
63. \( y' = y - 4x \), (2, 2)  
64. \( y' = y + xy \), (0, -4)  

65. **Slope Field** Use the slope field for the differential equation \( y' = 1/x \), where \( x > 0 \), to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of \( y' = 1/x \) as \( x \to \infty \). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

   (a) \( (1, 0) \)  
   (b) \( (2, -1) \)
66. Slope Field Use the slope field for the differential equation \( y' = \frac{1}{y} \), where \( y > 0 \), to sketch the graph of the solution that satisfies each initial condition. Then make a conjecture about the behavior of a particular solution of \( y' = \frac{1}{y} \) as \( x \to \infty \). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

(a) \((0, 1)\)  
(b) \((1, 1)\)

Slope Fields In Exercises 67–72, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.

67. \( \frac{dy}{dx} = 0.25y, \quad y(0) = 4 \)
68. \( \frac{dy}{dx} = 4 - y, \quad y(0) = 6 \)
69. \( \frac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2 \)
70. \( \frac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9 \)
71. \( \frac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1 \)
72. \( \frac{dy}{dx} = \frac{1}{2}e^{-x/4} \sin \frac{\pi x}{4}, \quad y(0) = 2 \)

Euler’s Method In Exercises 73–78, use Euler’s Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use \( n \) steps of size \( h \).

73. \( y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1 \)
74. \( y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05 \)
75. \( y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05 \)
76. \( y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4 \)
77. \( y' = e^x, \quad y(0) = 1, \quad n = 10, \quad h = 0.1 \)
78. \( y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1 \)

In Exercises 79–81, complete the table using the exact solution of the differential equation and two approximations obtained using Euler’s Method to approximate the particular solution of the differential equation. Use \( h = 0.2 \) and \( h = 0.1 \) and compute each approximation to four decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(x) ) (exact)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y(x) ) (( h = 0.2 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y(x) ) (( h = 0.1 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table for 79–81

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>79. ( \frac{dy}{dt} = y )</td>
<td>(0, 3)</td>
<td>( y = 3e^t )</td>
</tr>
<tr>
<td>80. ( \frac{dy}{dt} = \frac{2t}{y} )</td>
<td>(0, 2)</td>
<td>( y = \sqrt{2t^2 + 4} )</td>
</tr>
<tr>
<td>81. ( \frac{dy}{dt} = y + \cos(x) )</td>
<td>(0, 0)</td>
<td>( y = \frac{1}{2} (\sin x - \cos x + e^t) )</td>
</tr>
</tbody>
</table>

82. Compare the values of the approximations in Exercises 79–81 with the values given by the exact solution. How does the error change as \( h \) increases?

83. Temperature At time \( t = 0 \) minutes, the temperature of an object is 140°F. The temperature of the object is changing at the rate given by the differential equation

\[ \frac{dy}{dt} = -\frac{1}{2} (y - 72) \]

(a) Use a graphing utility and Euler’s Method to approximate the particular solutions of this differential equation at \( t = 1, 2, \) and 3. Use a step size of \( h = 0.1 \). (A graphing utility program for Euler’s Method is available at the website collegeject.com.)

(b) Compare your results with the exact solution

\[ y = 72 + 68e^{-t/2} \]

(c) Repeat parts (a) and (b) using a step size of \( h = 0.05 \). Compare the results.

CAPSTONE

84. The graph shows a solution of one of the following differential equations. Determine the correct equation. Explain your reasoning.

(a) \( y' = xy \)
(b) \( y' = \frac{4x}{y} \)
(c) \( y' = -4xy \)
(d) \( y' = 4 - xy \)
85. In your own words, describe the difference between a general solution of a differential equation and a particular solution.

86. Explain how to interpret a slope field.

87. Describe how to use Euler’s Method to approximate a particular solution of a differential equation.

88. It is known that $y = Ce^{2x}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine $C$ or $k$ from the information given? If so, find its value.

**True or False?** In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.

90. The general solution of a differential equation is $y = -4.2x^2 + C_1x + C_2$. To find a particular solution, you must be given two initial conditions.

91. Slope fields represent the general solutions of differential equations.

92. A slope field shows that the slope at the point $(1, 1)$ is 6. This slope field represents the family of solutions for the differential equation $y' = 4x + 2y$.

93. **Errors and Euler’s Method** The exact solution of the differential equation

$$\frac{dy}{dx} = -2y$$

where $y(0) = 4$, is $y = 4e^{-2x}$.

(a) Use a graphical utility to complete the table, where $y$ is the exact value of the solution, $y_1$ is the approximate solution using Euler’s Method with $h = 0.1$, $y_2$ is the approximate solution using Euler’s Method with $h = 0.2$, $e_1$ is the absolute error $|y - y_1|$, $e_2$ is the absolute error $|y - y_2|$, and $r$ is the ratio $e_1/e_2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$e_1$</td>
<td></td>
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<tr>
<td>$e_2$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$r$</td>
<td></td>
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</tbody>
</table>

(b) What can you conclude about the ratio $r$ as $h$ changes?

(c) Predict the absolute error when $h = 0.05$.

94. **Errors and Euler’s Method** Repeat Exercise 93 for which the exact solution of the differential equation

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$, is $y = x - 1 + 2e^{-x}$.

95. **Electric Circuits** The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor.

A model of the current $I$, in amperes (A), at time $t$ is given by the first-order differential equation

$$\frac{dI}{dt} + RI = E(t)$$

where $E(t)$ is the voltage (V) produced by the power source, $R$ is the resistance, in ohms (Ω), and $L$ is the inductance, in henrys (H). Suppose the electric circuit consists of a 24-V power source, a 12-Ω resistor, and a 4-H inductor.

(a) Sketch a slope field for the differential equation.

(b) What is the limiting value of the current? Explain.

96. **Think About It** It is known that $y = e^{kt}$ is a solution of the differential equation $y'' - 16y = 0$. Find the values of $k$.

97. **Think About It** It is known that $y = A \sin ax$ is a solution of the differential equation $y'' + 16y = 0$. Find the values of $a$.

**Putnam Exam Challenge**

98. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f'(x) = -xg(x)f(x)$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

99. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \quad p(x) \cdot q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

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Section 5.8

10. \[ \coth^2 x - \text{csch}^2 x = \frac{\cosh^2 x}{\sinh^2 x} - \frac{1}{\sinh^2 x} = \frac{\cosh^2 x - 1}{\sinh^2 x} \] Find Identity

26. \[ y = x \cosh(a) - \sinh(x) \]

40. \[ \int \frac{\cosh(\sqrt{x})}{\sqrt{x}} \, dx \quad u = x^{\frac{1}{2}} \]

63. \[ f(x) = \coth^{-1}(x^2) \] Look Up Formula

For \( \frac{dx}{dx} \{\coth^{-1}(a)\} \)

94. \[ \int \frac{dx}{(x+2) \sqrt{x^2 + 4x + 8}} \quad \text{Complete Square!} \]