Section 8.3

\[ \int \frac{-3x}{(x^2+3)^2} \, dx \]

2. For integrals involving \( \sqrt{a^2 + u^2} \), let

\[ u = a \tan \theta. \]

Then \( \sqrt{a^2 + u^2} = a \sec \theta \), where

\( -\pi/2 < \theta < \pi/2. \)

\[ U = x = \sqrt{3} \tan \theta \]

\[ dx = \sqrt{3} \sec^2 \theta \, d\theta \]

\[ \sqrt{x^2 + 3} = \sqrt{3} \sec \theta \]

Then Substitute

\[ \int -\frac{3x}{(x^2+3)^2} \, dx = \int \frac{-3(\sqrt{3} \tan \theta)(\sqrt{3} \sec^2 \theta \, d\theta)}{(\sqrt{3} \sec \theta)^3} \]

Convert to Sines and Cosines

\[ = \frac{-9 \tan \theta}{3 \sqrt{3} \sec \theta} \, d\theta \]

[Another Way]

\[ U = x^2 + 3 \]

\[ du = 2x \, dx \]

Use Power Rule

\[ = -\frac{3}{\sqrt{3}} \int \frac{\sin \theta}{\cos \theta} \, d\theta \]

\[ = -\sqrt{3} \int \sin \theta \, d\theta \]

\[ = -\sqrt{3} (-\cos \theta + C) \]

Use Triangle

\[ = \sqrt{3} \left( \frac{\sqrt{3}}{\sqrt{x^2+3}} \right) + C \]

\[ = \frac{3}{\sqrt{x^2+3}} + C \]
\[ \int \text{arcsec}(2x) \, dx \] Integrate by parts.

\[ u = \text{arcsec}(2x), \quad du = \frac{dx}{x \sqrt{x^2 - 1}} \]
\[ dv = dx, \quad v = x \]

\[ = u \cdot v - \int v \, du \]
\[ = x \cdot \text{arcsec}(2x) - \int \frac{x \, dx}{x \sqrt{x^2 - 1}} \]
\[ = x \cdot \text{arc sec}(2x) - \int \frac{dx}{\sqrt{4x^2 - 1}} \]

3. For integrals involving \( \sqrt{u^2 - a^2} \), let
\[ u = a \sec \theta, \quad \theta = \arccos \left( \frac{a}{u} \right) \]

Then
\[ \sqrt{u^2 - a^2} = \begin{cases} a \tan \theta & \text{if } u > a, \quad 0 \leq \theta < \pi/2 \\ -a \tan \theta & \text{if } u < -a, \quad \pi/2 < \theta \leq \pi \end{cases} \]

\[ a = 1 \]
\[ u = 2x = 1 \sec \theta \]
\[ x = \frac{1}{2} \sec \theta \]
\[ dx = \frac{1}{2} \sec \theta \tan \theta \, d\theta \]

\[ \int \frac{dx}{\sqrt{4x^2 - 1}} = \int \frac{1}{2} \sec \theta \tan \theta \, d\theta \]
\[ = \frac{1}{2} \, \sec \theta \tan \theta \, \text{sec \theta} + C \]
\[ = \frac{1}{2} \ln | \sec \theta + \tan \theta | + C \]

Use the triangle:
\[ = \frac{1}{2} \ln | 2x + \sqrt{4x^2 - 1} | + C \]

\[ x \cdot \text{arc sec}(2x) - \frac{1}{2} \ln | 2x + \sqrt{4x^2 - 1} | + C \]
3. For integrals involving \( \sqrt{u^2 - a^2} \), let 
\[ u = a \sec \theta, \]

Then 
\[ \sqrt{u^2 - a^2} = \begin{cases} 
    a \tan \theta & \text{if } u > a, \text{ where } 0 \leq \theta < \pi/2 \\
    -a \tan \theta & \text{if } u < -a, \text{ where } \pi/2 < \theta \leq \pi 
\end{cases} \]

\[ u = x = 5 \sec \theta \]
\[ dx = 5 \sec \theta \tan \theta \, d\theta \]

\[ \int x^3 \sqrt{x^2 - 25} \, dx = \int (5 \sec \theta)^3 (5 \tan \theta) (5 \sec \theta \tan \theta \, d\theta) \]
\[ = 5 \int 5 \sec^4 \theta \tan \theta \, d\theta \]
\[ = 3125 \int \sec^4 \theta \tan \theta \, d\theta \]

**GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT**

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.
TRIGONOMETRIC SUBSTITUTION ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$ 

Then $\sqrt{a^2 - u^2} = a \cos \theta$, where $-\pi/2 \leq \theta \leq \pi/2$.

$$u = t = a \sin \theta$$
$$dt = a \cos \theta \, d\theta$$

$$\sqrt{1-t^2} = a \cos \theta$$

$$\int \frac{(\sin \theta)^2 (a \cos \theta \, d\theta)}{(a \cos \theta)^3} = \int \frac{\sin \theta}{a \cos^2 \theta} = \int \tan^2 \theta \, d\theta$$

$t = 0, t = \frac{a}{2}$

$0 = \sin \theta$

$\frac{a}{2} = \sin \theta$

$\sin^{-1} \left( \frac{a}{2} \right) = \theta$

$0 = t$

$\frac{\pi}{2} = t$

**Convert Limits of Integration**

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

\[
\int \tan^n x \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx
\]

\[
= \sqrt{3} - \frac{\pi}{3}
\]

\[
0.685
\]
8.5 Partial Fractions

Understand the concept of partial fraction decomposition.
- Use partial fraction decomposition with linear factors to integrate rational functions.
- Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the method of partial fractions. To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6} \, dx.$$

To evaluate this integral without partial fractions, you can complete the square and use trigonometric substitution (see Figure 8.13) to obtain

$$\int \frac{dx}{(x - 5/2)^2 - (1/2)^2} = \int \frac{1}{(1/2) \sec \theta \tan \theta} d\theta = \int \frac{2 \csc \theta \, d\theta}{\tan \theta}$$

$$= \frac{\ln |\csc \theta - \cot \theta|}{2} + C$$

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

Partial fraction decomposition

Then you could evaluate the integral easily, as follows.

$$\int \frac{1}{x^2 - 5x + 6} \, dx = \int \left( \frac{1}{x - 3} - \frac{1}{x - 2} \right) \, dx$$

$$= \ln |x - 3| - \ln |x - 2| + C$$

This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator, $x^2 - 5x + 6$, and to find the partial fractions

$$\frac{1}{x - 3} \quad \text{and} \quad \frac{1}{x - 2}.$$

In this section, you will study techniques for finding partial fraction decompositions.
Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors. For instance, the polynomial
\[ x^5 + x^4 - x - 1 \]
can be written as
\[ x^5 + x^4 - x - 1 = x^4(x + 1) - (x + 1) \]
\[ = (x^4 - 1)(x + 1) \]
\[ = (x^2 + 1)(x^2 - 1)(x + 1) \]
\[ = (x^2 + 1)(x + 1)(x - 1)(x + 1) \]
\[ = (x - 1)(x + 1)^2(x^2 + 1) \]
where \( (x - 1) \) is a linear factor, \( (x + 1)^2 \) is a repeated linear factor, and \( (x^2 + 1) \) is an irreducible quadratic factor. Using this factorization, you can write the partial fraction decomposition of the rational expression
\[ \frac{N(x)}{x^5 + x^4 - x - 1} \]
where \( N(x) \) is a polynomial of degree less than 5, as follows.
\[ \frac{N(x)}{(x - 1)(x + 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{Dx + E}{x^2 + 1} \]
DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = \text{(a polynomial)} + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$\frac{A_1}{(px + q)^m} \quad \text{and} \quad \frac{A_2}{(ax^2 + bx + c)^n}$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of $m$ fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \ldots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of $n$ fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \ldots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$
Linear Factors

Algebraic techniques for determining the constants in the numerators of a partial fraction decomposition with linear or repeated linear factors are shown in Examples 1 and 2.

Try It 1

Write the partial fraction decomposition for \( \frac{3x^2 - 7x - 2}{x^3 - x} \).

\[
\frac{3x^2 - 7x - 2}{x^3 - x} = \frac{3x^2 - 7x - 2}{x(x^2 - 1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}
\]

**NOTE**: Note that the substitutions for \( x \) in Example 1 are chosen for their convenience in determining values for the unknown coefficients. While \( x = 2 \) is chosen to eliminate the term \( A(x - 2) \), and \( x = 3 \) is chosen to eliminate the term \( B(x - 3) \). The goal is to make convenient substitutions whenever possible.

\[
3x^2 - 7x - 2 = A(x-1)(x+1) + B(x-1) + C(x+1)
\]

Let \( x = 1 \)

\[
3(0)^2 - 7(0) - 2 = A(-1)(2) + B(0) + C(2)
\]

\[
-2 = -2C
\]

Let \( x = -1 \)

\[
3(-1)^2 - 7(-1) - 2 = A(-2)(0) + B(-1)(2) + C(-1)(0)
\]

\[
6 = 2B
\]

Let \( x = 0 \)

\[
3(0)^2 - 7(0) - 2 = A(-1)(0) + B(0)(-1) + C(0)(1)
\]

\[
-2 = -A - 0 + C
\]

\[
-2 = -A + 0 + C
\]

\[
-2 = -A + 0 + 0
\]

\[
-2 = -A
\]

\[
A = 2
\]

\[
\int \left( \frac{3x^2 - 7x - 2}{x^3 - x} \right) \, dx = \int \left( \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \right) \, dx
\]

\[
= 2 \ln |x| + 4 \ln |x+1| - 3 \ln |x-1| + x + \ln \left( \frac{x^2(x+1)^2}{(x-1)^3} \right) + C
\]
Try It 2

Evaluate \( \int \frac{2(x^2 - 4x - 15)}{x^3 + 3x^2 - 9x - 27} \, dx \).

\[
2x^2 - 8x - 30 = \frac{A}{x-3} + \frac{B}{x+3} + \frac{C}{(x+3)^2}
\]

\[
\begin{align*}
L C D &= (x+3)^2 (x-3) \\
L C D &= (x+3)^2 (x-3)
\end{align*}
\]

Let \( x = 3 \)

\[
2(3)^2 - 8(3) - 30 = A(3)^2 + B(3)(3-3) + C(3-3)
\]

\[
36 = 36A
\]

\[
A = 1
\]

Let \( x = -3 \)

\[
2(-3)^2 - 8(-3) - 30 = A(0)^2 + B(0)(-3) + C(-3)
\]

\[
12 = -6C
\]

\[
C = -2
\]

Let \( A = -1 \) and \( C = -2 \) and \( x = 0 \)

\[
2(0)^2 - 8(0) - 30 = (-1)(0)^2 + B(0)(0) + (-2)(-3)
\]

\[
-30 = -9 - 9B + 6
\]

\[
-30 = -9B - 3
\]

\[
27 = 9B
\]

\[
B = 3
\]

\[
\int \frac{2x^2 - 8x - 30}{(x-3)(x+3)^2} \, dx = \int \left[ \frac{-1}{x-3} + \frac{3}{x+3} - \frac{2}{(x+3)^2} \right] \, dx
\]

\[
-\ln |x-3| + 3\ln |x+3| + \frac{2}{x+3} + C
\]

\[
\text{NOTE: It is necessary to make as many substitutions for } x \text{ as there are unknowns (A, B, C, . . .) to be determined. For instance, in Example 2, three substitutions (x = 0, x = -1, and x = 1) were made to solve for A, B, and C.}
\]
Quadratic Factors

When using the method of partial fractions with linear factors, a convenient choice of $x$ immediately yields a value for one of the coefficients. With quadratic factors, a system of linear equations usually has to be solved, regardless of the choice of $x$.

Try It 3

Evaluate $\int \frac{x}{x^3 - 1} \, dx$. 
Try It 4

Evaluate \[ \int \frac{2x^2 + x + 8}{(x^2 + 4)^2} \, dx. \]

\[
\begin{align*}
\frac{2x^2 + x + 8}{(x^2 + 4)^2} &= \frac{Ax + B}{(x^2 + 4)^1} + \frac{Cx + D}{(x^2 + 4)^2} \\
&= \frac{(Ax + B)(x^2 + 4) + Cx + D}{(x^2 + 4)^2} \\
&= \frac{Ax^3 + 4Ax + Bx^2 + 4B + Cx + D}{(x^2 + 4)^2}
\end{align*}
\]

\[
\begin{align*}
0 &= A \\
2 &= B \\
1 &= 4A + C \\
8 &= 4B + D \\
8 &= 8 + D \\
0 &= 0
\end{align*}
\]

\[
\begin{align*}
\int \frac{2x^2 + x + 8}{(x^2 + 4)^2} \, dx &= \int \frac{0x^2}{(x^2 + 4)^1} \, dx + \int \frac{(1x + 0)}{(x^2 + 4)^2} \, dx \\
&= \int \frac{2}{x^2} \, dx + \frac{1}{2} \int \frac{2x}{(x^2 + 4)^2} \, dx \\
&= \frac{1}{2} \int (x^2 + 4)^{-2}(2x \, dx) \\
&= 2 \cdot \frac{1}{2} \arctan \left( \frac{x}{2} \right) + \frac{1}{2} \left( \frac{x^2 + 1}{-1} \right)^{-1} + C \\
&= \arctan \left( \frac{x}{2} \right) - \frac{1}{2(x^2 + 1)} + C
\end{align*}
\]
When integrating rational expressions, keep in mind that for *improper* rational expressions such as
\[
\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}
\]
you must first divide to obtain
\[
\frac{N(x)}{D(x)} = 2x - 1 + \frac{-2x + 5}{x^2 + x - 2}
\]
The proper rational expression is then decomposed into its partial fractions by the usual methods. Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

**GUIDELINES FOR SOLVING THE BASIC EQUATION**

**Linear Factors**
1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of \( x \) and solve for the remaining coefficients.

**Quadratic Factors**
1. Expand the basic equation.
2. Collect terms according to powers of \( x \).
3. Equate the coefficients of like powers to obtain a system of linear equations involving \( A, B, C \), and so on.
4. Solve the system of linear equations.
Before concluding this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

\[
\int \frac{x^2 + 1}{x^3 + 3x - 4} \, dx = \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x - 4} \, dx \\
= \frac{1}{3} \ln |x^3 + 3x - 4| + C
\]

Second, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

\[
\int \frac{x^2 - x - 2}{x^3 - 2(x + 2)} \, dx = \int \frac{(x + 1)(x - 2)}{(x - 2)(x^2 + 2x + 2)} \, dx \\
= \int \frac{x + 1}{x^2 + 2x + 2} \, dx \\
= \frac{1}{2} \ln |x^2 + 2x + 2| + C
\]

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution \( u = \sin x \) allows you to write

\[
\int \frac{\cos x}{\sin x (\sin x - 1)} \, dx = \int \frac{du}{u(u - 1)} \quad u = \sin x, \ du = \cos x \, dx
\]
In Exercises 1–6, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

1. \[ \frac{2x + 1}{x^2 + 4} \]
2. \[ \frac{x + 3}{x^2 + 1} \]
3. \[ \frac{3x - 2}{x^2 + 2} \]
4. \[ \frac{x - 4}{x^2 + 6x + 5} \]
5. \[ \frac{x - 9}{x^2 - 6x} \]
6. \[ \frac{2x - 1}{x^2 + 3} \]

In Exercises 7–28, use partial fractions to find the integral.

7. \[ \int \frac{1}{x^2 - 9} \, dx \]
8. \[ \int \frac{1}{4x^2 - 1} \, dx \]
9. \[ \int \frac{5}{x^3 + 3x - 7} \, dx \]
10. \[ \int \frac{2x}{x^2 - 4} \, dx \]
11. \[ \int \frac{2x}{x^3 - 4} \, dx \]
12. \[ \int \frac{x + 2}{x^2 + 1} \, dx \]
13. \[ \int \frac{1}{x^3 + 2x} \, dx \]
14. \[ \int \frac{3x^2 + x}{x^3 + x^2 + 2} \, dx \]
15. \[ \int \frac{x^2 - 4x - 10}{x^2 - 2x - 8} \, dx \]
16. \[ \int \frac{x^2}{x^3 - 4} \, dx \]
17. \[ \int \frac{x^2 + 2x - 1}{x^3 + 3} \, dx \]
18. \[ \int \frac{3x - 4}{x^2 - 2x - 8} \, dx \]
19. \[ \int \frac{x^2 + 3x - 4}{x^3 - 4x} \, dx \]
20. \[ \int \frac{4}{x^2 + x + 2} \, dx \]
21. \[ \int \frac{x}{x + 1} \, dx \]
22. \[ \int \frac{6x}{x^2 - 1} \, dx \]

In Exercises 29–32, evaluate the definite integral. Use a graphing utility to verify your result.

29. \[ \int_{0}^{1} \frac{2x^2 + 5x + 1}{x^2 + 1} \, dx \]
30. \[ \int_{1}^{2} \frac{x - 1}{x(x + 1)} \, dx \]
31. \[ \int_{0}^{1} \frac{1}{x^2 + x + 1} \, dx \]
32. \[ \int_{0}^{1} \frac{2x - 1}{x^2 + x + 1} \, dx \]

In Exercises 33–40, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.

33. \[ \int \frac{5x}{x^3 - 10x + 25} \, dx \], \( (6, 0) \)
34. \[ \int \frac{6x^2 + 1}{x(x^2 - 1)} \, dx \], \( (2, 1) \)
35. \[ \int \frac{x^2 + x + 2}{x^3 + 2x^2} \, dx \], \( (0, 1) \)
36. \[ \int \frac{x^2}{(x^2 - 4)^2} \, dx \], \( (3, 4) \)
37. \[ \int \frac{x^2 - 2x + 3}{x^3 - x^2 - x - 2} \, dx \], \( (3, 10) \)
38. \[ \int \frac{1}{x(x^2 - 9)} \, dx \], \( (3, 2) \)
39. \[ \int \frac{1}{x^2 - x} \, dx \], \( (7, 2) \)
40. \[ \int \frac{x^2 - x + 2}{x^3 - x^2 - x - 1} \, dx \], \( (2, 6) \)

In Exercises 41–50, use substitution to find the integral.

41. \[ \int \frac{\sin x}{\cos x + \cos x - 1} \, dx \]
42. \[ \int \frac{\sin x}{\cos x + \cos x} \, dx \]
43. \[ \int \frac{\cos x}{\sin x + \sin x} \, dx \]
44. \[ \int \frac{5 \cos x}{\sin x + \sin x} \, dx \]
45. \[ \int \frac{\tan x + 5 \tan x + 6}{\sec x} \, dx \]
46. \[ \int \frac{\tan x + 5 \tan x + 6}{\sec x} \, dx \]
47. \[ \int \frac{e^x}{e^x + 1} \, dx \]
48. \[ \int \frac{e^x}{e^x + 1} \, dx \]
49. \[ \int \frac{1}{\sqrt{x} - \sqrt{x}} \, dx \]
50. \[ \int \frac{1}{\sqrt{x} - \sqrt{x}} \, dx \]

In Exercises 51–54, use the method of partial fractions to verify the integration formula.

51. \[ \int \frac{1}{a(a + b)} \, dx = \frac{1}{a} \ln |a + b| + C \]
52. \[ \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln \left| \frac{x}{a - x} \right| + C \]
53. \[ \int \frac{x}{(a + b)x^2} \, dx = \frac{1}{a} \ln \left| \frac{x}{a + b} \right| + C \]
54. \[ \int \frac{1}{x^2(a + bx)} \, dx = \frac{1}{ax} \ln \left| \frac{x}{a + bx} \right| + C \]

**CAS: Slope Fields** In Exercises 55 and 56, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the given initial condition.

55. \[ \frac{dy}{dx} = \frac{6}{4 - x^2} \]
56. \[ \frac{dy}{dx} = \frac{4}{x^2 - 2x - 3} \]

**WRITING ABOUT CONCEPTS**

57. What is the first step when integrating \( \int \frac{x^3}{x - 5} \, dx \)? Explain.

58. Describe the decomposition of the proper rational function \( N(x)/D(x) \) if \( D(x) = (px + q)^2 \) and \( (a^2 + bx + c)^2 \), where \( a^2 + bx + c \) is irreducible. Explain why you chose that method.
59. **Area** Find the area of the region bounded by the graphs of 
\[ y = \frac{12}{x^2 + 5x + 6}, \quad y = 0, \quad x = 0, \text{ and } x = 1. \]

60. **Area** Find the area of the region bounded by the graphs of 
\[ y = \frac{15}{x^2 + 7x + 12}, \quad y = 0, \quad x = 0, \text{ and } x = 2. \]

61. **Area** Find the area of the region bounded by the graphs of 
\[ y = \frac{7}{16 - x^2} \text{ and } y = 1. \]

---

**CAPSTONE**

62. State the method you would use to evaluate each integral. Explain why you chose that method. Do not integrate.

(a) \[ \int \frac{x + 1}{x^2 + 2x - 8} \, dx \]

(b) \[ \int \frac{7x + 4}{x^2 + 2x - 8} \, dx \]

(c) \[ \int \frac{4}{x^2 + 2x + 5} \, dx \]

63. **Modeling Data** The predicted cost \( C \) (in hundreds of thousands of dollars) for a company to remove \( p\% \) of a chemical from its waste water is shown in the table.

<table>
<thead>
<tr>
<th>( p )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>0.7</td>
<td>1.0</td>
<td>1.3</td>
<td>1.7</td>
<td>2.0</td>
<td>2.7</td>
<td>3.6</td>
<td>5.5</td>
<td>11.2</td>
<td></td>
</tr>
</tbody>
</table>

A model for the data is given by \( C = \frac{124p}{(10 + p)(100 - p)} \), \( 0 \leq p < 100 \). Use the model to find the average cost of removing between 75\% and 80\% of the chemical.

64. **Logistic Growth** In Chapter 6, the exponential growth equation was derived from the assumption that the rate of growth was proportional to the existing quantity. In practice, there often exists some upper limit \( L \) past which growth cannot occur. In such cases, you assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity \( y \) and the upper limit \( L \). That is, \( \frac{dy}{dt} = ky(L - y) \). In integral form, you can write this relationship as

\[
\int \frac{dy}{y(L - y)} = \int k \, dt.
\]

(a) A slope field for the differential equation \( \frac{dy}{dt} = y(3 - y) \) is shown. Draw a possible solution to the differential equation if \( y(0) = 5 \), and another if \( y(0) = \frac{1}{2} \). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

(b) Where \( y(0) \) is greater than 3, what is the sign of the slope of the solution?

(c) For \( y > 0 \), find \( \lim_{x \to \infty} y(t) \).

(d) Evaluate the two given integrals and solve for \( y \) as a function of \( t \), where \( y_0 \) is the initial quantity.

(e) Use the result of part (d) to find and graph the solutions in part (a).

(f) The graph of the function \( y \) is a **logistic curve**. Show that the rate of growth is maximum at the point of inflection, and that this occurs when \( y = L/2 \).

65. **Volume and Centroid** Consider the region bounded by the graphs of \( y = 2x/(x^2 + 1) \), \( y = 0 \), \( x = 0 \), and \( x = 3 \). Find the volume of the solid generated by revolving the region about the \( x \)-axis. Find the centroid of the region.

66. **Volume** Consider the region bounded by the graph of \( y^2 = (2 - x)^2/(1 + x)^2 \) on the interval \([0, 1]\). Find the volume of the solid generated by revolving this region about the \( x \)-axis.

67. **Epidemic Model** A single infected individual enters a community of \( n \) susceptible individuals. Let \( x \) be the number of newly infected individuals at time \( t \). The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So, \( dx/dt = k(x + 1)(n - x) \) and you obtain

\[
\int \frac{1}{(x + 1)(n - x)} \, dx = \int k \, dt.
\]

Solve for \( x \) as a function of \( t \).

68. **Chemical Reactions** In a chemical reaction, one unit of compound \( Y \) and one unit of compound \( Z \) are converted into a single unit of compound \( X \). \( x \) is the amount of compound \( X \) formed, and the rate of formation of \( X \) is proportional to the product of the amounts of unconverted compounds \( Y \) and \( Z \). So, \( dx/dt = k(y_0 - x)(z_0 - x) \), where \( y_0 \) and \( z_0 \) are the initial amounts of compounds \( Y \) and \( Z \). From this equation you obtain

\[
\int \frac{1}{(y_0 - x)(z_0 - x)} \, dx = \int k \, dt.
\]

(a) Perform the two integrations and solve for \( x \) in terms of \( t \).

(b) Use the result of part (a) to find \( x \) as \( t \to \infty \) if (1) \( y_0 < z_0 \), (2) \( y_0 > z_0 \), and (3) \( y_0 = z_0 \).

69. Evaluate

\[
\int_0^1 \frac{x}{1 + x^2} \, dx
\]

in two different ways, one of which is partial fractions.

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**PUTNAM EXAM CHALLENGE**

70. Prove \( \frac{22}{7} - \pi = \int_0^1 \frac{x^4(1 - x)^4}{1 + x^2} \, dx \).

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.