\[
\int \frac{x^2 + s}{x^3 - x^2 + x + 3} \, dx
\]

**Synthetic Division**

Multiply by LCD

\[
\left[ \frac{x^2 + s}{(x+1)(x^2 - 2x + 3)} \right] = \frac{A}{x+1} + \frac{Bx+C}{x^2 - 2x + 3}
\]

Let \(x = -1\)

\[
6 = 6A \quad \text{or} \quad A = 1
\]

Equate Coefficients

\[
x^2 + s = A(x^2 - 2x + 3) + (Bx+C)(x+1)
\]

\[
l \begin{array}{c}
1 & 1 & 3 \\
2 & 1 & 1 & 3 & 9 \\
1 & 1 & 6 & 1 & 4 \\
-11 & 1 & -2 & 3 & 0
\end{array}
\]

\[
\frac{(x-1)}{(x+1)(1x^2 - 2x + 3)}
\]

\[
I \text{ know that } A = 1
\]

\[
1 = A + B \\
0 = -2A + B + C \\
5 = 3A + C
\]

\[
S = 3(1) + C = 2 = C
\]

\[
S \frac{x^2 + s}{x^3 - x^2 + x + 3} \, dx = S \frac{1}{x+1} \, dx + S \frac{2x + 2}{x^2 - 2x + 3} \, dx
\]

\[
\overrightarrow{(x-1)^2 + 2}
\]

\[
S \frac{1}{x+1} \, dx + S \frac{2}{(x-1)^2 + (\sqrt{2})^2} \, dx
\]

\[
u = x - 1, \quad a = \sqrt{2}
\]

\[
\ln |x+1| + 2 \cdot \frac{1}{\sqrt{2}} \arctan \left( \frac{x-1}{\sqrt{2}} \right) + c
\]
8.6 Integration by Tables and Other Integration Techniques

- Evaluate an indefinite integral using a table of integrals.
- Evaluate an indefinite integral using reduction formulas.
- Evaluate an indefinite integral involving rational functions of sine and cosine.

Integration by Tables

So far in this chapter you have studied several integration techniques that can be used with the basic integration rules. But merely knowing how to use the various techniques is not enough. You also need to know when to use them. Integration is first and foremost a problem of recognition. That is, you must recognize which rule or technique to apply to obtain an antiderivative. Frequently, a slight alteration of an integrand will require a different integration technique (or produce a function whose antiderivative is not an elementary function), as shown below.

\[
\begin{align*}
\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \\
\int \frac{\ln x}{x} \, dx &= \frac{(\ln x)^2}{2} + C \\
\int \frac{1}{x \ln x} \, dx &= \ln|\ln x| + C \\
\int \frac{x}{\ln x} \, dx &= ?
\end{align*}
\]

Integration by parts

Power Rule

Log Rule

Not an elementary function

Many people find tables of integrals to be a valuable supplement to the integration techniques discussed in this chapter. Tables of common integrals can be found in Appendix B. Integration by tables is not a “cure-all” for all of the difficulties that can accompany integration—using tables of integrals requires considerable thought and insight and often involves substitution.

Each integration formula in Appendix B can be developed using one or more of the techniques in this chapter. You should try to verify several of the formulas. For instance, Formula 4

\[
\int \frac{u}{(a + bu)^2} \, du = \frac{1}{b^2} \left( \frac{a}{a + bu} + \ln|a + bu| \right) + C \quad \text{Formula 4}
\]

can be verified using the method of partial fractions, and Formula 19

\[
\int \frac{a + bu}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}} \quad \text{Formula 19}
\]

can be verified using integration by parts. Note that the integrals in Appendix B are classified according to forms involving the following.

\[
\begin{align*}
u^n & \quad (a + bu) \\
(a + bu + cu^2) & \quad \sqrt{a + bu} \\
(a^2 \pm u^2) & \quad \sqrt{u^2 \pm a^2} \\
\sqrt{a^2 - u^2} & \quad \text{Trigonometric functions} \\
\text{Inverse trigonometric functions} & \quad \text{Exponential functions} \\
\text{Logarithmic functions}
\end{align*}
\]
Try It 1

Evaluate \( \int \frac{dx}{2x^2(2x-1)^2} \)

Use Tables

\[ a = -1 \quad u = x \quad du = dx \]
\[ b = 2 \]

\[ \frac{1}{2} \int \frac{dx}{x^2(2x-1)^2} = \left( \frac{1}{2} \right) \frac{-1}{(c-1)^2} \left[ \frac{-1+4x}{x(c-1+2x)} \right] - 4 \ln \left| \frac{x}{c-2x} \right| + C \]

\[ \frac{1}{2} \left[ \frac{4x-1}{x(2x-1)} - 4 \ln \left| \frac{x}{2x-1} \right| \right] + C \]
Try It 2

Evaluate \[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx \]

\[ u = x, \quad a = 2 \]
\[ du = dx \]

\[ \int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a \arccsc \frac{u}{a} + C \]

\[ = \sqrt{x^2 - 4} - 2 \arccsc \left[ \frac{1 \cdot x}{2} \right] + C \]
Try It 3

Evaluate $\int e^{-2x} \cos 3x \, dx$.

$q = -2 \quad \frac{e^{-2x}}{\sqrt{2^2 + 3^2}} \left[ -2 \cos (3x) + 3 \sin (3x) \right] + C$

$b = 3 \quad \frac{e^{-2x} \cos (3x) + b \sin (3x)}{a^2 + b^2} + C$

Table 86

$\int e^u \cos bu \, du = \frac{e^u}{a^2 + b^2} (a \cos bu + b \sin bu) + C$
Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x) \, dx = g(x) + \int h(x) \, dx$. Such integration formulas are called reduction formulas because they reduce a given integral to the sum of a function and a simpler integral.

Try It 4

Consider the following two formulas.

\[
\Rightarrow \int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du \quad \text{Formula 54}
\]

\[
\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du \quad \text{Formula 55}
\]

Evaluate $\int t^4 \cos t \, dt$.

Let $u = t$, $n = 4$

$du = dt$

\[
\int t^4 \cos t \, dt = \int u^4 \cos u \, du = \int (t^4) \cos t \, dt = \int (u^4) \cos u \, du
\]

\[
= t^4 \sin t - 4 \left[ \int t^3 \cos t \, dt \right]
\]

\[
= t^4 \sin t - 4 \left[ -t^3 \sin t + 3 \int t^2 \cos t \, dt \right]
\]

\[
= t^4 \sin t + 4t^3 \cos t - 12 \left[ \int t^2 \cos t \, dt \right]
\]

\[
= t^4 \sin t + 4t^3 \cos t - 12 \left[ 2t \sin t + \int 2 \cos t \, dt \right]
\]

\[
= t^4 \sin t + 4t^3 \cos t - 12t \sin t - 24 \left[ t \sin t + \int 2 \cos t \, dt \right]
\]

\[
= t^4 \sin t + 4t^3 \cos t - 12t^2 \sin t - 24 \sin t + 24 \cos t + C
\]

\[
\int u \sin u \, du = \sin u - u \cos u + C
\]
Try It 5

Evaluate $\int \sec^5 x \, dx$.

Consider the following two formulas.

\[
\int \sec u \, du = \ln|\sec u + \tan u| + C
\]

\[
\int \sec^n u \, du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} u \, du, \quad n \neq 1
\]

Formula 61

Formula 69

\[
\#67 = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x \, dx
\]

\[
\#69 = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \left[ \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \right]
\]

\[
\#67 = \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \tan x + \frac{3}{8} \left[ \ln |\sec x + \tan x| \right] + C
\]

\[
\#69 = \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \tan x + \frac{3}{8} \left[ \ln |\sec x + \tan x| \right] + C
\]
Try It 6

Evaluate \( \int \frac{\sin x}{1 + \sin x} \, dx \).

Solution

\[
\int \frac{\sin x}{1 + \sin x} \, dx = \int \frac{\sin x}{1 + \sin x} \left( \frac{1 - \sin x}{1 - \sin x} \right)
\]

\[
= \frac{\sin x - \sin^2 x}{1 - \sin^2 x} = \frac{\sin x - \sin^2 x}{\cos^2 x}
\]

Multiply by Conjugate 1

\[
= \frac{\sin x - \sin^2 x}{\cos^2 x} = \frac{\sin x}{\cos x} - \frac{\sin^2 x}{\cos^2 x}
\]

\[
= \int \frac{\sin x}{\cos x} \, dx - \int \frac{\sin^2 x}{\cos^2 x} \, dx
\]

\[
= \int \csc x \, dx - \int \tan x \, dx
\]

\[
= \ln |\csc x - \cot x| + C
\]

Alternatively:

\[
\int (\cos x)^{-1} \sin x \, dx = \int (\cos x)^{-1} u^{-2} \, du
\]

\[
= \int \frac{u^{-1}}{u^{-2}} \, du = \int u \, du
\]

\[
= \frac{1}{2} u^2 + C
\]

\[
= \frac{1}{2} \cos x + C
\]

\[
= \frac{1}{2} \sec x + C
\]

\[
\int \tan x \, dx
\]

\[
= \ln |\sec x + x - \tan x + C|
\]
Example 6 involves a rational expression of $\sin x$ and $\cos x$. If you are unable to find an integral of this form in the integration tables, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

### SUBSTITUTION FOR RATIONAL FUNCTIONS OF SINE AND COSINE

For integrals involving rational functions of sine and cosine, the substitution

\[
  u = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}
\]

yields

\[
  \cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2\, du}{1 + u^2}.
\]

**Proof**

From the substitution for $u$, it follows that

\[
u^2 = \frac{\sin^2 x}{(1 + \cos x)^2} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{1 - \cos x}{1 + \cos x}.
\]

Solving for $\cos x$ produces $\cos x = (1 - u^2)/(1 + u^2)$. To find $\sin x$, write $u = \sin x/(1 + \cos x)$ as

\[
\sin x = u(1 + \cos x) = u \left( 1 + \frac{1 - u^2}{1 + u^2} \right) = \frac{2u}{1 + u^2}.
\]

Finally, to find $dx$, consider $u = \tan(x/2)$. Then you have $\arctan u = x/2$ and $dx = (2\, du)/(1 + u^2)$. 

### Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

\[ u = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2} \]

yields

\[ \cos x = \frac{1 - u^2}{1 + u^2}, \sin x = \frac{2u}{1 + u^2}, \text{ and } dx = \frac{2\, du}{1 + u^2}. \]

\[
\int_{\theta}^{\pi/2} \frac{1}{3 - 2\cos \theta} \, d\theta = \int_{0}^{1} \left[ \frac{2\, du}{3 - 2 \left( \frac{1 - u^2}{1 + u^2} \right)} \right]
\]

\[
= \int_{0}^{1} \frac{2\, du}{1 + u^2} \cdot \frac{1}{\frac{3(1 + u^2) - 2(1 - u^2)}{1 + u^2}} = \int_{0}^{1} \frac{2\, du}{1 + u^2}
\]

\[
= \frac{2}{\sqrt{5}} \arctan \frac{u}{\sqrt{5}} + C
\]

\[
= \frac{2}{\sqrt{5}} \arctan \left( \frac{a}{\sqrt{5}} \right) + C
\]

\[
= \frac{2}{\sqrt{5}} \arctan (a) - 0
\]

\[
= \frac{2}{\sqrt{5}} \arctan (\cos \theta)
\]
\[ \int_{0}^{2} \sqrt{16 - 4x^2} \, dx \]

\[ 2\pi \]

Trig Substitution:

\[ a = 4 \]
\[ u = 2x = 4 \sin \theta \]
\[ \sin \theta = \frac{2x}{4} \]
\[ \sqrt{16 - 4x^2} = 4 \cos \theta \]

\[ x = 0 \quad \theta = 0 \]
\[ \theta = \frac{\pi}{2} \]

\[ \sin \theta = \frac{2(\theta)}{4} \]
\[ \sin \theta = \frac{2(\theta)}{4} \]

\[ \sin \theta = 1 \quad \theta = \frac{\pi}{2} \]
### 8.6 Integration by Tables and Other Integration Techniques

#### Exercises


In Exercises 1 and 2, use a table of integrals with forms involving \( a + bu \) to find the integral.

1. \( \int \frac{x^3}{5 + x} \, dx \)
2. \( \int \frac{2}{3x^3(2x - 5)} \, dx \)

In Exercises 3 and 4, use a table of integrals with forms involving \( \sqrt{u^2 - a^2} \) to find the integral.

3. \( \int e^{\sqrt{1 + e^3}} \, dx \)
4. \( \int \frac{\sqrt{x^3 - 3x}}{6x} \, dx \)

In Exercises 5 and 6, use a table of integrals with forms involving \( \sqrt{a^2 - u^2} \) to find the integral.

5. \( \int \frac{1}{x^2 \sqrt{1 - x^2}} \, dx \)
6. \( \int \frac{\sqrt{x^3 - 6x - 10}^2}{x} \, dx \)

In Exercises 7–10, use a table of integrals with forms involving the trigonometric functions to find the integral.

7. \( \int \cos^4 3x \, dx \)
8. \( \int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} \, dx \)
9. \( \int \frac{1}{\sqrt{5} \sqrt{1 - \cos x}} \, dx \)
10. \( \int \frac{1}{1 - \tan 5x} \, dx \)

In Exercises 11 and 12, use a table of integrals with forms involving \( e^x \) to find the integral.

11. \( \int \frac{1}{1 + e^x} \, dx \)
12. \( \int e^{-x^2} \sin 2x \, dx \)

In Exercises 13 and 14, use a table of integrals with forms involving \( \ln u \) to find the integral.

13. \( \int x^7 \ln x \, dx \)
14. \( \int (\ln x)^3 \, dx \)

In Exercises 15–18, find the indefinite integral (a) using integration tables and (b) using the given method.

<table>
<thead>
<tr>
<th>Integral</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>15. ( \int x^3 e^{x^4} , dx )</td>
<td>Integration by parts</td>
</tr>
<tr>
<td>16. ( \int e^x \ln x , dx )</td>
<td>Integration by parts</td>
</tr>
<tr>
<td>17. ( \int \frac{1}{x^2(x + 1)} , dx )</td>
<td>Partial fractions</td>
</tr>
<tr>
<td>18. ( \int \frac{1}{x^3 - 48} , dx )</td>
<td>Partial fractions</td>
</tr>
</tbody>
</table>

In Exercises 19–42, use integration tables to find the integral.

19. \( \int x \arccsc(x^2 + 1) \, dx \)
20. \( \int \arccsc 2x \, dx \)
21. \( \int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx \)
22. \( \int \frac{1}{x^3 + 4x + 8} \, dx \)
23. \( \int \frac{4x}{(2 - 5x)^2} \, dx \)
24. \( \int \frac{\theta^2}{1 - \sin \theta} \, d\theta \)
25. \( \int e^x \arccos e^x \, dx \)
26. \( \int e^x \tan^{-1} e^x \, dx \)
27. \( \int \frac{x}{1 - \sec x} \, dx \)
28. \( \int \frac{1}{1 + (ln x)^2} \, dx \)
29. \( \int \frac{\cos \theta}{3 + 2 \sin \theta + \sin^2 \theta} \, d\theta \)
30. \( \int x^2 \sqrt{2 + 9x^2} \, dx \)
31. \( \int \frac{x}{x^2 + 2 + 9x^2} \, dx \)
32. \( \int x \arctan x^{1/2} \, dx \)
33. \( \int \frac{x}{x(3 + 2 \ln x)} \, dx \)
34. \( \int \frac{e^x}{(1 - e^{2x})^{1/2}} \, dx \)
35. \( \int \frac{x}{(x^2 - 6x + 10)} \, dx \)
36. \( \int (2x - 3)^2 \sqrt{2x - 3} + 4 \, dx \)
37. \( \int \frac{x}{\sqrt{x^4 - 6x^5 + 5}} \, dx \)
38. \( \int \frac{\cos x}{\sqrt{\sin^2 x + 1}} \, dx \)
39. \( \int \frac{x^3}{\sqrt{4 - x^2}} \, dx \)
40. \( \int \frac{\sqrt{x - x}}{\sqrt{5 + x}} \, dx \)
41. \( \int e^{2x} \, dx \)
42. \( \int \cot^4 \theta \, d\theta \)

In Exercises 43–50, use integration tables to evaluate the integral.

43. \( \int_0^x x e^x \, dx \)
44. \( \int_0^x \frac{x e^x}{\sqrt{9 + x}} \, dx \)
45. \( \int_0^x \frac{x}{\cos x} \, dx \)
46. \( \int_0^x \frac{e^x}{\cos x} \, dx \)
47. \( \int_0^{\pi/2} \frac{x}{1 + \sin^2 x} \, dx \)
48. \( \int_0^{\pi/2} \frac{x}{(2x - \pi)^2} \, dx \)
49. \( \int_0^1 t^5 \cos t \, dt \)
50. \( \int_0^1 \sqrt{3 + x^2} \, dx \)

In Exercises 51–56, verify the integration formula.

51. \( \int \frac{u^2}{(a + bu)^2} \, du = \frac{1}{b^2} \left( bu - \frac{a^2}{a + bu} - 2a \ln |a + bu| \right) + C \)
52. \( \int \frac{u^2}{\sqrt{a + bu}} \, du = \frac{2}{2n + 1} \left( u^n \sqrt{a + bu} - nu \right) + C \)
53. \( \int \frac{1}{(a^2 + b^2)^{1/2}} \, du = \frac{\pm a}{a^2 \sqrt{a^2 + b^2}} + C \)
54. \( \int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du \)
55. \( \int u \arctan u \, du = u \arctan u - \ln \sqrt{1 + u^2} + C \)
56. \( \int (\ln u)^n \, du = u(\ln u)^n - n \int (\ln u)^{n-1} \, du \)
In Exercises 57–62, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.

57. \( \int \frac{1}{x^2\sqrt{1-x}} \, dx \), \( (\frac{1}{2}, 5) \)
58. \( x\sqrt{x^2 + 2x} \, dx \), \( (0, 0) \)
59. \( \int \frac{1}{(x^2 - 6x + 10)^3} \, dx \), \( (3, 0) \)
60. \( \int \frac{1}{x + 1} \, dx \), \( (0, \sqrt{2}) \)
61. \( \int \frac{1}{\sin \theta \tan \theta} \, d\theta \), \( (\frac{\pi}{4}, 2) \)
62. \( \int \frac{1}{\cos \theta (1 + \sin \theta)} \, d\theta \), \( (0, 1) \)

In Exercises 63–70, find or evaluate the integral.

63. \( \int \frac{1}{2 - 3 \sin \theta} \, d\theta \)
64. \( \int \frac{\sin \theta}{1 + \cos^2 \theta} \, d\theta \)
65. \( \int_0^\pi \frac{1}{1 + \sin \theta + \cos \theta} \, d\theta \)
66. \( \int_0^\pi \frac{1}{3 - 2 \cos \theta} \, d\theta \)
67. \( \int \frac{\sin \theta}{\cos \theta} \, d\theta \)
68. \( \int \frac{\cos \theta}{1 + \cos \theta} \, d\theta \)
69. \( \int \frac{\cos \theta}{\sin \theta} \, d\theta \)
70. \( \int \frac{4}{\csc \theta - \cot \theta} \, d\theta \)

Area In Exercises 71 and 72, find the area of the region bounded by the graphs of the equations.

71. \( y = \frac{x}{\sqrt{x} + 3} \), \( x = 0, x = 6 \)
72. \( y = \frac{1}{1 + e^{-x}} \), \( y = 0, x = 2 \)

**Writing About Concepts**

73. (a) Evaluate \( \int x^n \, dx \) for \( n = 1, 2, 3 \). Describe any patterns you notice.
    (b) Write a general rule for evaluating the integral in part (a), for an integer \( n \geq 1 \). 

74. Describe what is meant by a reduction formula. Give an example.

**True or False?** In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

75. To use a table of integrals, the integral you are evaluating must appear in the table.
76. When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

77. **Volume** Consider the region bounded by the graphs of \( y = x^2 / 16 - x^2, y = 0, x = 0, \) and \( x = 4 \). Find the volume of the solid generated by revolving the region about the y-axis.

**CAPSTONE**

78. State (if possible) the method or integration formula you would use to find the antiderivative. Explain why you chose that method or formula. Do not integrate.

(a) \( \int e^{x^2 + 1} \, dx \)
(b) \( \int e^{-x} \, dx \)
(c) \( \int x e^{3x} \, dx \)
(d) \( \int e^{x^2} \, dx \)
(e) \( \int e^{2x} \, dx \)
(f) \( \int e^{3x} + 1 \, dx \)

79. **Work** A hydraulic cylinder on an industrial machine pushes a steel block a distance of \( x \) feet \((0 \leq x \leq 5)\), where the variable force required is \( F(x) = 20000e^{-x} \) pounds. Find the work done in pushing the block the full 5 feet through the machine.

80. **Work** Repeat Exercise 79, using \( F(x) = -\frac{50000}{\sqrt{26 - x^2}} \) pounds.

81. **Building Design** The cross section of a precast concrete beam for a building is bounded by the graphs of the equations

\[ x = \frac{2}{\sqrt{1 + y^2}}, y = \frac{x}{\sqrt{1 + y^2}}, y = 0, \text{ and } y = 3 \]

where \( x \) and \( y \) are measured in feet. The length of the beam is 20 feet (see figure). (a) Find the volume \( V \) and the weight \( W \) of the beam. Assume the concrete weighs 148 pounds per cubic foot. (b) Then find the centroid of a cross section of the beam.

**PUTNAM EXAM CHALLENGE**

85. Evaluate \( \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{1/2}} \).

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