Section 9.7

5th Degree Maclaurin Polynomial

\[ f(x) = \frac{1}{x+1} = (x+1)^{-1} \quad f(0) = 1 \]

\[ f'(x) = -\frac{1}{(x+1)^2} \quad f'(0) = -1 \]

\[ f''(x) = 2 \frac{1}{(x+1)^3} \quad f''(0) = 2 \]

\[ f'''(x) = -6 \frac{1}{(x+1)^4} \quad f'''(0) = -6 \]

\[ f^{(4)}(x) = 24 \frac{1}{(x+1)^5} \quad f^{(4)}(0) = 24 \]

\[ f^{(5)}(x) = -120 \frac{1}{(x+1)^6} \quad f^{(5)}(0) = -120 \]

\[ = \frac{f(0)}{0!} + \frac{f'(0)(x-0)}{1!} + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \ldots \]

\[ P_5(x) : 1 - x + \frac{x^2}{2} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \ldots \]

\[ P_5(x) : 1 - x + x^2 - x^3 + x^4 - x^5 \]

5th Degree Maclaurin Polynomial

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

\[ P_5(0.3) = 1 - (0.3) + (0.3)^2 - (0.3)^3 + (0.3)^4 - (0.3)^5 \]

\[ = 0.7687 \]

\[ f(0.3) = \frac{1}{x+1} = \frac{1}{1.3} = 0.76923 \]
\[ f(x) = \ln x \quad \text{centered at} \ x = 2 \]

Taylor Polynomial:

\[ P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]

\[ P_1(x) = \ln 2 + \frac{1}{2}(x-2) \]

\[ P_2(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4}(x-2)^2 + \frac{1}{6}(x-2)^3 \]

\[ P_3(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4}(x-2)^2 + \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 \]

\[ P(2.5) = \ln 2 + \frac{1}{2}(2.5) - \frac{1}{4}(2.5)^2 + \frac{1}{6}(2.5)^3 - \frac{1}{12}(2.5)^4 \]

\[ P(2.5) \approx 0.83296 \]

\[ f(2.5) \approx \ln(2.5) = 0.83290 \] Calculated

**Theorem 5.19: Taylor's Theorem**

If a function \( f \) is differentiable through order \( n + 1 \) in an interval \( I \) containing \( c \), then for each \( x \) in \( I \), there exists \( z \) between \( x \) and \( c \) such that

\[ f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x) \]

where

\[ R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \]

Note: One useful consequence of Taylor's Theorem is that

\[ R_n(c) = \frac{f^{(n+1)}(z)}{(n+1)!}(c-c)^{n+1} = 0 \]

where \( \max |f^{(n+1)}(z)| \) is the maximum value of \( |f^{(n+1)}(z)| \) between \( x \) and \( c \).

\[ |R_n(x)| \leq \frac{M}{(n+1)!} \max |f^{(n+1)}(z)| \]

For \( f(2.5) = 2x^2 - 6x^4 \):

\[ f''(2) = 24x \]

\[ f''(2) = 24(2) = 48 \]

Estimated error:

\[ |R_n(x)| \leq \frac{M}{(n+1)!} \max |f^{(n+1)}(z)| \]

**Example:**

\[ f(2.5) \approx 2 \quad \text{when} \quad \frac{29}{2} = 2 \frac{9}{2} = 5 \]
$f(x) = \sqrt{x}$

$f(6) = \sqrt{6}$