Section 9.9

(17) $h(x) = \frac{-2}{x^2 - 1} = \frac{-2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$\frac{2}{1-x^2} = \frac{-2}{(x+1)(x-1)} = \frac{1}{1+x} + \frac{1}{1-x}$

$\frac{1}{1-x} = \frac{a}{1-r}$ where $a = 1$, $r = x$

$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 1 = (x)^n$

$\frac{1}{1+r} = \frac{1}{1-(x)} = \frac{9}{1-r}$ where $a = 1$, $r = -(x)$

$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 1 = (-x)^n$

$\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left( (-1)^n + 1 \right) x^n$

$I = c = (-\infty, 1)$

Geometric Series
19 \frac{d}{dx} \left[ \frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}

\frac{1}{1-(x)} = \frac{1}{1-x}

\sum_{n=0}^{\infty} (-1)^n x^n =

\frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] =

\sum_{n=0}^{\infty} (-1)^n \cdot n \cdot x^{n-1}

x = 1

\sum_{n=1}^{\infty} (-1)^n \cdot n \cdot x^{n-1} \rightarrow 

Diverge

By n-th Term Test.

\sum_{n=1}^{\infty} (G)^n (-1)^{n-1} \rightarrow 

Odd \n \rightarrow 

Neg.

\sum_{n=1}^{2n-1} (n) \rightarrow

Satisfies nth Term Test. S. Diverges
\[ \arctan \left( \frac{1}{y} \right) = \sum_{n=0}^{\infty} \left( -\frac{1}{y^2} \right)^n \frac{2n+1}{2n+1} \]

\[ = \frac{1}{y} - \frac{1}{3y^3} + \frac{1}{7y^5} + \frac{1}{11y^7} + \frac{1}{15y^9} + \frac{1}{19y^{11}} + \frac{1}{23y^{13}} + \frac{1}{27y^{15}} + \frac{1}{31y^{17}} + \dots \]

\[ |R_N| \leq a_{n+1} \quad |R_2| < a_3 \]

\[ 3rd = \frac{1}{5120} < 0.001 \quad \frac{1}{4} - \frac{1}{192} \approx 0.245 \]

\[ \text{arctan } x = \int \frac{1}{1 + x^2} \, dx + C \]

\[ = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \]

\[ \boxed{\text{arctan } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} \]

\[ = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. \]

Let \( x = 0 \), then \( C = 0 \).

Interval of convergence: \([-1, 1]\).
Taylor and Maclaurin Series

In Section 9.9, you derived power series for several functions using geometric series with term-by-term differentiation or integration. In this section you will study a general procedure for deriving the power series for a function that has derivatives of all orders. The following theorem gives the form that every convergent power series must take:

**THEOREM 9.22 THE FORM OF A CONVERGENT POWER SERIES**

If $f$ is represented by a power series $f(x) = \sum a_n(x - c)^n$ for all $x$ in an open interval $I$ containing $c$, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

**Proof** Suppose the power series $\sum a_n(x - c)^n$ has a radius of convergence $R$. Then, by Theorem 9.21, you know that the $n$th derivative of $f$ exists for $|x - c| < R$, and by successive differentiation you obtain the following.

- $f^{(0)}(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + a_4(x - c)^4 + \cdots$
- $f^{(1)}(x) = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + \cdots$
- $f^{(2)}(x) = 2a_2 + 3a_3(x - c) + 4a_4(x - c)^2 + \cdots$
- $f^{(3)}(x) = 3a_3 + 4a_4(x - c) + \cdots$
- $f^{(n)}(x) = n!a_n + (n + 1)!a_{n+1}(x - c) + \cdots$

Evaluating each of these derivatives at $x = c$ yields

- $f^{(0)}(c) = a_0$
- $f^{(1)}(c) = 0!a_1$
- $f^{(2)}(c) = 2!a_2$
- $f^{(3)}(c) = 3!a_3$

and, in general, $f^{(n)}(c) = n!a_n$. By solving for $a_n$, you find that the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Notice that the coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at $c$ as defined in Section 9.7. For this reason, the series is called the Taylor series for $f(x)$ at $c$.

**NOTE** Be sure you understand Theorem 9.22. The theorem says that if a power series converges to $f(x)$, the series must be a Taylor series. The theorem does not say that every series formed with the Taylor coefficients $a_n = f^{(n)}(c)/n!$ will converge to $f(x)$.
DEFINITION OF TAYLOR AND MACLAURIN SERIES

If a function \( f \) has derivatives of all orders at \( x = c \), then the series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots
\]

is called the Taylor series for \( f(x) \) at \( c \). Moreover, if \( c = 0 \), then the series is the Maclaurin series for \( f \).

If you know the pattern for the coefficients of the Taylor polynomials for a function, you can extend the pattern easily to form the corresponding Taylor series. For instance, in Example 4 in Section 9.7, you found the fourth Taylor polynomial for \( \ln x \), centered at 1, to be:

\[
P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \cdots.
\]

From this pattern, you can obtain the Taylor series for \( \ln x \) centered at \( c = 1 \):

\[
(x - 1) - \frac{1}{2}(x - 1)^2 + \cdots + \frac{(-1)^{n+1}}{n}(x - 1)^n + \cdots.
\]

**EXAMPLE 1**  

**Forming a Power Series**

Use the function \( f(x) = \sin x \) to form the Maclaurin series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots
\]

and determine the interval of convergence.

**Solution**  

Successive differentiation of \( f(x) \) yields

\[
\begin{align*}
f(x) &= \sin x & f(0) &= \sin 0 = 0 & \longrightarrow 0 \\
f'(x) &= \cos x & f'(0) &= \cos 0 = 1 & \longrightarrow 1 \\
f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 & \longrightarrow 0 \\
f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 & \longrightarrow -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 & \longrightarrow 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1 & \longrightarrow 1
\end{align*}
\]

and so on. The pattern repeats after the third derivative. So, the power series is as follows.

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\]

By the Ratio Test, you can conclude that this series converges for all \( x \).
Try It 1

Use the function \( f(x) = e^{-2x} \) to form the Maclaurin series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots
\]

and determine the interval of convergence.

\[
f(x) = e^{-2x} \\
f'(x) = -2e^{-2x} \\
f''(x) = 4te^{-2x} \\
f'''(x) = -8e^{-2x} \\
f''''(x) = 16e^{-2x} \\
f^{(5)}(x) = -32e^{-2x}
\]

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots
\]

\[
= 1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} - \frac{32x^5}{5!} + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n
\]

General Term

of Maclaurin Series

Ratio Test

\[
\lim_{n \to \infty} \left| \frac{2^{n+1} x^{n+1} \cdot \frac{1}{(n+1)!}}{(n+1)! \cdot \frac{2^n x^n}{n!}} \right| = \lim_{n \to \infty} \left| \frac{2x}{n+1} \right|
\]

\[
= \lim_{n \to \infty} \left| \left( \frac{2x}{n+1} \right) \left( \frac{1}{n+1} \right) \right|
\]

\[
= \lim_{n \to \infty} \left| \left( \frac{2x}{n+1} \right) \cdot (0) \right|
\]

\[
= 0 < 1
\]

Converge for All Real \#s

By Ratio Test
Notice that in Example 1 you cannot conclude that the power series converges to \( \sin x \) for all \( x \). You can simply conclude that the power series converges to some function, but you are not sure what function it is. This is a subtle, but important, point in dealing with Taylor or Maclaurin series. To persuade yourself that the series

\[
f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots
\]

might converge to a function other than \( f \), remember that the derivatives are being evaluated at a single point. It can easily happen that another function will agree with the values of \( f^{(n)}(x) \) when \( x = c \) and disagree at other \( x \)-values. For instance, if you formed the power series (centered at 0) for the function shown in Figure 9.23, you would obtain the same series as in Example 1. You know that the series converges for all \( x \), and yet it obviously cannot converge to both \( f(x) \) and \( \sin x \) for all \( x \).

Let \( f \) have derivatives of all orders in an open interval \( I \) centered at \( c \). The Taylor series for \( f \) may fail to converge for some \( x \) in \( I \). Or, even if it is convergent, it may fail to have \( f(x) \) as its sum. Nevertheless, Theorem 9.19 tells us that for each \( n \),

\[
f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),
\]

where

\[
R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - c)^{n+1},
\]

Note that in this remainder formula the particular value of \( \xi \) that makes the remainder formula true depends on the values of \( x \) and \( n \). If \( R_n \to 0 \), then the following theorem tells us that the Taylor series for \( f \) actually converges to \( f(x) \) for all \( x \) in \( I \).
**THEOREM 9.23 CONVERGENCE OF TAYLOR SERIES**

If \( \lim_{n \to \infty} R_n = 0 \) for all \( x \) in the interval \( I \), then the Taylor series for \( f \) converges and equals \( f(x) \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.
\]

**PROOF** For a Taylor series, the \( n \)th partial sum coincides with the \( n \)th Taylor polynomial. That is, \( S_n(x) = P_n(x) \). Moreover, because

\[
P_n(x) = f(x) - R_n(x)
\]

it follows that

\[
\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} P_n(x)
= \lim_{n \to \infty} [f(x) - R_n(x)]
= f(x) - \lim_{n \to \infty} R_n(x).
\]

So, for a given \( x \), the Taylor series (the sequence of partial sums) converges to \( f(x) \) if and only if \( R_n(x) \to 0 \) as \( n \to \infty \).

**NOTE** Stated another way, Theorem 9.23 says that a power series formed with Taylor coefficients \( a_n = \frac{f^{(n)}(c)}{n!} \) converges to the function from which it was derived at precisely those values for which the remainder approaches 0 as \( n \to \infty \).
In Example 1, you derived the power series from the sine function and you also concluded that the series converges to some function on the entire real line. In Example 2, you will see that the series actually converges to \( \sin x \). The key observation is that although the value of \( z \) is not known, it is possible to obtain an upper bound for \( |f^{(n+1)}(z)| \).

**Try It 2**

Show that the Maclaurin series for \( f(x) = e^x \) converges for all \( x \).

**Solution**

You need to show that

\[
e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots
\]

is true for all \( x \). Because \( f^{(n+1)}(x) = e^x \)

you know that \( 0 < |f^{(n+1)}(z)| \) for every real number \( z \). Therefore, for any fixed \( x \), you can apply Taylor's Theorem (Theorem 9.19) to conclude that

\[
0 < |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{e^x x^{n+1}}{(n+1)!} \right| = \left| e^x - \frac{x^{n+1}}{(n+1)!} \right|
\]

From the discussion in Section 9.1 regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed \( x \)

\[
\lim_{n \to \infty} \frac{|x|^{n+1}}{(n + 1)!} = 0.
\]

Finally, by the Squeeze Theorem, it follows that for all \( x \), \( R_n(x) \to 0 \) and \( n \to \infty \). Hence, by Theorem 9.23, the Maclaurin series for \( e^x \) converges to \( e^x \) for all \( x \).

\[
0 < \left| \frac{e^x \cdot x^{n+1}}{(n+1)!} \right| = 0
\]
Figure 9.24 visually illustrates the convergence of the Maclaurin series for \( \sin x \) by comparing the graphs of the Maclaurin polynomials \( P_1(x) \), \( P_3(x) \), \( P_5(x) \), and \( P_7(x) \) with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.

As \( n \) increases, the graph of \( P_n \) more closely resembles the sine function. 

Figure 9.24
The guidelines for finding a Taylor series for $f(x)$ at $c$ are summarized below.

**GUIDELINES FOR FINDING A TAYLOR SERIES**

1. Differentiate $f(x)$ several times and evaluate each derivative at $c$.
   \[
   f(c), f'(c), f''(c), f'''(c), \ldots, f^{(n)}(c), \ldots
   \]
   Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients
   \[ a_n = \frac{f^{(n)}(c)}{n!}, \]
   and determine the interval of convergence for the resulting power series
   \[
   f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots
   \]

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.
   \[
   \text{Check that } R_n(x) \to 0
   \]

The direct determination of Taylor or Maclaurin coefficients using successive differentiation can be difficult, and the next example illustrates a shortcut for finding the coefficients indirectly—using the coefficients of a known Taylor or Maclaurin series.
Try It 3

Find the Maclaurin series for \( f(x) = \cos 3x \)

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} \quad \cdots \quad -\infty < x < \infty
\]

Substitute \( 3x \) for \( x \)

\[
\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \cdots
\]

\[
\cos(3x) = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!}
\]

Use short cuts
Use Table of Sines

General Term
Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form \( f(x) = (1 + x)^k \). This produces the binomial series.

**EXAMPLE 4 Binomial Series**

Find the Maclaurin series for \( f(x) = (1 + x)^k \) and determine its radius of convergence. Assume that \( k \) is not a positive integer.

**Solution**  
By successive differentiation, you have

\[
\begin{align*}
    f(x) &= (1 + x)^k & f(0) &= 1 \\
    f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
    f''(x) &= k(k - 1)(1 + x)^{k-2} & f''(0) &= k(k - 1) \\
    f'''(x) &= k(k - 1)(k - 2)(1 + x)^{k-3} & f'''(0) &= k(k - 1)(k - 2) \\
    & \vdots \ \\
    f^{(n)}(x) &= k \cdots (k - n + 1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k - 1) \cdots (k - n + 1)
\end{align*}
\]

which produces the series:

\[
1 + kx + \frac{k(k - 1)x^2}{2} + \cdots + \frac{k(k - 1) \cdots (k - n + 1)x^n}{n!} + \cdots.
\]

Because \( a_{n+1}/a_n \to 1 \), you can apply the Ratio Test to conclude that the radius of convergence is \( R = 1 \). So, the series converges to some function in the interval \((-1, 1)\).

Note that Example 4 shows that the Taylor series for \( (1 + x)^k \) converges to some function in the interval \((-1, 1)\). However, the example does not show that the series actually converges to \( (1 + x)^k \). To do this, you could show that the remainder \( R(x) \) converges to 0, as illustrated in Example 2.
Try It 5

Find the power series for \( f(x) = \sqrt{1 + x} \).

\[
= (1 + x)^{\frac{1}{2}}
\]

Solution

Using the binomial series

\[
(1 + x)^k = 1 + kx + \frac{k(k - 1)x^2}{2!} + \frac{k(k - 1)(k - 2)x^3}{3!} + \ldots
\]

Let \( k = \frac{1}{2} \) and write

\[
= 1 + \frac{1}{2}x + \frac{(\frac{1}{2})x - \frac{1}{2})x^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \ldots
\]

\[
= 1 + \frac{1}{2}x - \frac{x^2}{2^2 2!} + \frac{3x^3}{2^3 3!} - \frac{1 \cdot 3 \cdot 5x^4}{2^4 4!} + \ldots
\]

\[
(-1, 1)
\]

Check End.\[\]
POWER SERIES FOR ELEMENTARY FUNCTIONS

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \ldots + (-1)^n (x - 1)^n + \ldots$</td>
<td>$0 &lt; x &lt; 2$</td>
</tr>
<tr>
<td>$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots + (-1)^n x^n + \ldots$</td>
<td>$-1 &lt; x &lt; 1$</td>
</tr>
<tr>
<td>$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \ldots + \frac{(-1)^{n-1}(x - 1)^n}{n} + \ldots$</td>
<td>$0 &lt; x \leq 2$</td>
</tr>
<tr>
<td>$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
</tr>
<tr>
<td>$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!} + \ldots$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
</tr>
<tr>
<td>$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
</tr>
<tr>
<td>$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} + \ldots$</td>
<td>$-1 \leq x \leq 1$</td>
</tr>
<tr>
<td>$\arcsin x = x + \frac{x^3}{3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \ldots + \frac{(2n)! x^{2n+1}}{(2n)! (2n+1)} + \ldots$</td>
<td>$-1 \leq x \leq 1$</td>
</tr>
<tr>
<td>$(1 + x)^k = 1 + kx + \frac{k(k - 1)x^2}{2!} + \frac{k(k - 1)(k - 2)x^3}{3!} + \frac{k(k - 1)(k - 2)(k - 3)x^4}{4!} + \ldots$</td>
<td>$-1 &lt; x &lt; 1^*$</td>
</tr>
</tbody>
</table>

*The convergence at $x = \pm 1$ depends on the value of $k$."

NOTE The binomial series is valid for noninteger values of $k$. Moreover, if $k$ happens to be a positive integer, the binomial series reduces to a simple binomial expansion.
Try It 6

Find the power series for \( f(x) = \sin \sqrt{x} \).

Solution

Using the power series

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots
\]

\[
\sqrt{x} = x^{\frac{1}{2}} = \frac{x}{2} - \frac{3x^2}{8} + \frac{5x^3}{16} - \frac{7x^4}{32} + \ldots
\]

\[
\sin \sqrt{x} = x^{\frac{1}{2}} - \frac{\left(\frac{x^2}{2}\right)^3}{3!} + \frac{\left(\frac{x^2}{2}\right)^5}{5!} - \frac{\left(\frac{x^2}{2}\right)^7}{7!} + \frac{\left(\frac{x^2}{2}\right)^9}{9!}
\]

\[
= x^{\frac{1}{2}} - \frac{x^3}{2 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!}
\]
Power series can be multiplied and divided like polynomials. After finding the first few terms of the product (or quotient), you may be able to recognize a pattern.

**Try It 7**

Find the first three nonzero terms in the Maclaurin series \( \ln(x + 1) \arctan x \).

**Solution**

Using the Maclaurin series for \( \ln(x + 1) \) and \( \arctan x \), you have

\[
\ln(x + 1) \arctan x = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right) \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \right)
\]

Multiply these expressions and collect like terms as you would for multiplying polynomials

\[
\begin{align*}
\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \\
\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^6}{5} - \ldots \\
\hline
\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} - \ldots \\
- \frac{x^4}{3} + \frac{x^5}{6} - \frac{x^6}{9} + \ldots \\
\hline
\frac{x^2}{2} - \frac{x^3}{2} + 0 - \frac{2x^5}{24} + \frac{13x^6}{45} - \ldots
\end{align*}
\]

So, \( \ln(x + 1) \arctan x = x^2 - \frac{x^3}{2} - \frac{2x^5}{24} + \ldots \).
Try It 8

Find the power series for \( f(x) = \frac{1}{2} (e^x + e^{-x}) = \cosh x \).

Solution

Use the series for \( e^x \).

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots
\]

\[
e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots
\]

\[
e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \cdots
\]

\[
\cosh(x) = \frac{1}{2} (e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots
\]

This series converges for \(-\infty < x < \infty\).
Try It 9

Use a power series to approximate

\[
\int_0^{1/2} \frac{\ln(x + 1)}{x} \, dx
\]

with an error less than 0.0001.

Solution

Replacing \( x \) with \( x + 1 \) in the series for \( \ln x \) and then dividing by \( x \) produces the following.

\[
\frac{\ln(x + 1)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots
\]

\[
\int_0^{1/2} \frac{\ln(x + 1)}{x} \, dx = \left[ x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots \right]_0^{1/2}
\]

\[
= \frac{1}{2} - \frac{1}{2^22^2} + \frac{1}{3^22^3} - \frac{1}{4^22^4} + \frac{1}{5^22^5} - \frac{1}{6^22^6} + \frac{1}{7^22^7} - \frac{1}{8^22^8} + \cdots
\]

Summing the first seven terms, you have

\[
\int_0^{1/2} \frac{\ln(x + 1)}{x} \, dx \approx 0.4483
\]

which, by the Alternating Series Test, has an error of less than

\[
\frac{1}{8^22^8} \approx 0.00006.
\]
In Exercises 1–12, use the definition of Taylor series to find the Taylor series (centered at $c$) for the function.

1. $f(x) = e^{3x}, \quad c = 0$
2. $f(x) = e^{4x}, \quad c = 0$
3. $f(x) = \cos x, \quad c = \frac{\pi}{4}$
4. $f(x) = \sin x, \quad c = \frac{\pi}{4}$
5. $f(x) = \frac{1}{x}, \quad c = 1$
6. $f(x) = \frac{1}{1 - x}, \quad c = 2$
7. $f(x) = \ln x, \quad c = 1$
8. $f(x) = e^x, \quad c = 1$
9. $f(x) = \sin 3x, \quad c = 0$
10. $f(x) = \ln(x^2 + 1), \quad c = 0$
11. $f(x) = \sec x, \quad c = 0$ (first three nonzero terms)
12. $f(x) = \tan x, \quad c = 0$ (first three nonzero terms)

In Exercises 13–16, prove that the Maclaurin series for the function converges to the function for all $x$.

13. $f(x) = \cos x$
14. $f(x) = e^{-3x}$
15. $f(x) = \sinh x$
16. $f(x) = \cosh x$

In Exercises 17–26, use the binomial series to find the Maclaurin series for the function.

17. $f(x) = \frac{1}{(1 + x)^2}$
18. $f(x) = \frac{1}{(1 + x)^3}$
19. $f(x) = \frac{1}{\sqrt{1 - x}}$
20. $f(x) = \frac{1}{\sqrt{1 - x^2}}$
21. $f(x) = \frac{1}{\sqrt{4 + x^2}}$
22. $f(x) = \frac{1}{(2 + x)^3}$
23. $f(x) = \sqrt{1 + x}$
24. $f(x) = \sqrt{1 + x^2}$
25. $f(x) = \sqrt{1 + x^3}$
26. $f(x) = \sqrt{1 + x^4}$

In Exercises 27–40, find the Maclaurin series for the function. (Use the table of power series for elementary functions.)

27. $f(x) = e^{x/2}$
28. $g(x) = e^{-3x}$
29. $f(x) = \ln(1 + x)$
30. $f(x) = \ln(1 + x^2)$
31. $g(x) = \sin 3x$
32. $f(x) = \sin x$
33. $f(x) = \cos 4x$
34. $f(x) = \cos x$
35. $f(x) = \cos x^{5/2}$
36. $g(x) = 2 \sin x$
37. $f(x) = \frac{2}{3}(e^x - e^{-x}) = \sinh x$
38. $f(x) = e^x + e^{-x} = 2 \cosh x$
39. $f(x) = \cos^2 x$
40. $f(x) = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

(Hint: Integrate the series for $\frac{1}{\sqrt{x^2 + 1}}$)

In Exercises 41–44, find the Maclaurin series for the function. (See Example 7.)

41. $f(x) = x \sin x$
42. $h(x) = x \cos x$
43. $g(x) = \begin{cases} \sin x, & x \neq 0 \\ 1, & x = 0 \end{cases}$
44. $f(x) = \begin{cases} \frac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

In Exercises 45 and 46, use a power series and the fact that $i^2 = -1$ to verify the formula.

45. $g(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$
46. $g(x) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

In Exercises 47–52, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 684. Use a graphing utility to graph the function and its corresponding polynomial approximation.

47. $f(x) = e^x \sin x$
48. $g(x) = e^x \cos x$
49. $h(x) = \sin x \ln(1 + x)$
50. $f(x) = e^x \ln(1 + x)$
51. $g(x) = \frac{\sin x}{1 + x}$
52. $f(x) = \frac{e^x}{1 + x}$

In Exercises 53–56, match the polynomial with its graph. (The graphs are labeled (a), (b), (c), and (d).) Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.

53. $y = x^2 - \frac{x^4}{3!}$
54. $y = x^3 + \frac{x^5}{2!} + \frac{x^7}{4!}$
55. $y = x + x^2 + \frac{x^3}{2!}$
56. $y = x^3 - x + x^3 + x^4$
In Exercises 57 and 58, find a Maclaurin series for \( f(x) \).

57. \( f(x) = \int_0^x (e^{-t^2} - 1) \, dt \)

58. \( f(x) = \int_0^x \frac{1}{\sqrt{1 + t^4}} \, dt \)

In Exercises 59–62, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.

59. \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2 \)

60. \( \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{(2n + 1)!} \right] = \sin 1 \)

61. \( \sum_{n=0}^{\infty} \frac{1}{(2n)!} = e \)

62. \( \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n!} \right) = e - 1 \)

In Exercises 63–66, use the series representation of the function \( f \) to find \( \lim_{x \to 0} f(x) \) (if it exists).

63. \( f(x) = \frac{1 - \cos x}{x} \)  

64. \( f(x) = \frac{\sin x}{x} \)  

65. \( f(x) = \frac{\ln (x + 1)}{x} \)  

66. \( f(x) = \frac{\ln (x + 1)}{x} \)

In Exercises 67–74, use a power series to approximate the value of the integral with an error of less than 0.0001. (In Exercises 69 and 71, assume that the integrand is defined as 1 when \( x = 0 \).)

67. \( \int_0^1 e^{-x^2} \, dx \)  

68. \( \int_0^{1/4} x \ln(x + 1) \, dx \)

69. \( \int_0^1 \frac{\sin x}{x} \, dx \)  

70. \( \int_0^1 \cos x^2 \, dx \)

71. \( \int_0^{1/2} \frac{\arctan x}{x} \, dx \)  

72. \( \int_0^{1/2} \arctan x^2 \, dx \)

73. \( \int_0^{0.3} \sqrt{1 + x^3} \, dx \)  

74. \( \int_0^{0.2} \sqrt{1 + x^3} \, dx \)

In Exercises 75 and 76, use a power series to approximate the area of the region. Use a graphing utility to verify the result.

75. \( \int_0^{\pi/2} \sqrt{x} \cos x \, dx \)

76. \( \int_0^{\pi/5} \cos \sqrt{x} \, dx \)

Probability In Exercises 77 and 78, approximate the normal probability with an error of less than 0.0001, where the probability is given by

\[ P(a < x < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, dx. \]

77. \( P(0 < x < 1) \)

78. \( P(1 < x < 2) \)

In Exercises 79–82, use a computer algebra system to find the fifth-degree Taylor polynomial (centered at \( c \)) for the function. Graph the function and the polynomial. Use the graph to determine the largest interval on which the polynomial is a reasonable approximation of the function.

79. \( f(x) = x \cos 2x, \quad c = 0 \)

80. \( f(x) = \sin \frac{x}{2} \ln(1 + x), \quad c = 0 \)

81. \( g(x) = \sqrt{x} \ln x, \quad c = 1 \)

82. \( h(x) = \sqrt{x} \arctan x, \quad c = 1 \)

Writing about Concepts

83. State the guidelines for finding a Taylor series.

84. If \( f \) is an even function, what must be true about the coefficients \( a_n \) in the Maclaurin series

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]

Explain your reasoning.

85. Define the binomial series. What is its radius of convergence?

Capstone

86. Explain how to use the series

\[ g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

to find the series for each function. Do not find the series.

(a) \( f(x) = e^{-x} \)

(b) \( f(x) = e^{2x} \)

(c) \( f(x) = xe^x \)

(d) \( f(x) = e^{2x} + e^{-2x} \)
87. Projectile Motion A projectile fired from the ground follows the trajectory given by

\[ y = \left( \frac{\tan \theta - \frac{g}{kv_0 \cos \theta}}{k^2} \right) x - \frac{g}{k^2} \ln \left( 1 - \frac{kv}{v_0 \cos \theta} \right) \]

where \( v_0 \) is the initial speed, \( \theta \) is the angle of projection, \( g \) is the acceleration due to gravity, and \( k \) is the drag factor caused by air resistance. Using the power series representation

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1 \]

verify that the trajectory can be rewritten as

\[ y = (\tan \theta) x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{kgx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2 gx^4}{4v_0^4 \cos^4 \theta} + \cdots. \]

88. Projectile Motion Use the result of Exercise 87 to determine the series for the path of a projectile launched from ground level at an angle of \( \theta = 60^\circ \), with an initial speed of \( v_0 = 64 \) feet per second and a drag factor of \( k = \frac{1}{15} \).

89. Investigation Consider the function \( f \) defined by

\[ f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases} \]

(a) Sketch a graph of the function.

(b) Use the alternative form of the definition of the derivative (Section 2.1) and L'Hôpital's Rule to show that \( f'(0) = 0 \).

(c) Using the result in part (b), find the Maclaurin series for \( f \).

(d) Does the series converge to \( f \)?

90. Investigation

(a) Find the power series centered at 0 for the function

\[ f(x) = \frac{\ln(x^2 + 1)}{x^2} \]

(b) Use a graphing utility to graph \( f \) and the eighth-degree Taylor polynomial \( P_8(x) \) for \( f \).

(c) Complete the table, where

\[ F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt \quad \text{and} \quad G(x) = \int_0^x P_8(t) dt. \]

\[ \begin{array}{c|c|c|c|c|c|c|c} \hline x & 0.25 & 0.50 & 0.75 & 1.00 & 1.50 & 2.00 \\ \hline F(x) & \hline G(x) & \hline \end{array} \]

(d) Describe the relationship between the graphs of \( f \) and \( P_8 \) and the results given in the table in part (c).

91. Prove that

\[ \lim_{n \to \infty} x^n = 0 \quad \text{for any real} \ x. \]

92. Find the Maclaurin series for

\[ f(x) = \ln \frac{1 + x}{1 - x} \]

and determine its radius of convergence. Use the first four terms of the series to approximate \( \ln 3 \).

In Exercises 93–96, evaluate the binomial coefficient using the formula

\[ \left( \begin{array}{c} k \\ n \end{array} \right) = \frac{k(k-1)(k-2)(k-3) \cdots (k-n+1)}{n!} \]

where \( k \) is a real number, \( n \) is a positive integer, and

\[ \left( \begin{array}{c} k \\ n \end{array} \right) = 1. \]

93. \( \left( \begin{array}{c} 5 \\ 3 \end{array} \right) \), 94. \( \left( \begin{array}{c} -2 \\ 2 \end{array} \right) \), 95. \( \left( \begin{array}{c} 0.5 \\ 4 \end{array} \right) \), 96. \( \left( \begin{array}{c} -1/3 \\ 5 \end{array} \right) \)

97. Write the power series for \( (1 + x)^n \) in terms of binomial coefficients.

98. Prove that \( e \) is irrational. \( \text{\textit{Hint:}} \text{ Assume that } e = \frac{p}{q} \text{ is rational } (p \text{ and } q \text{ are integers}) \text{ and consider } e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \]

99. Show that the Maclaurin series for the function

\[ g(x) = \frac{x}{1 - x - x^3} \]

is

\[ \sum_{n=1}^{\infty} F_n x^n \]

where \( F_n \) is the \( n \)th Fibonacci number with \( F_1 = F_2 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 3 \).

(\text{\textit{Hint:}} \text{ Write } \frac{x}{1-x-x^2} = a_0 + a_1 x + a_2 x^2 + \cdots \text{ and multiply each side of this equation by } 1 - x = x^2.)

**PUTNAM EXAM CHALLENGE**

100. Assume that \( |f(x)| \leq 1 \) and \( |f^{(n)}(x)| \leq 1 \) for all \( x \) on an interval of length at least \( 2 \). Show that \( |f(x)| \leq 2 \) on the interval.

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.