In Exercises 45–72, determine the convergence or divergence of the sequence with the given $n$th term. If the sequence converges, find its limit.

45. $a_n = (0.3^n - 1$

46. $a_n = 4 - \frac{3}{n}$

47. $a_n = \frac{5}{n + 2}$

48. $a_n = \frac{2}{n}$

49. $a_n = (-1)^n \left( \frac{n}{n + 1} \right)$

50. $a_n = 1 + (-1)^n$

51. $a_n = \frac{3n^2 - n + 4}{2n^2 + 1}$

52. $a_n = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$

53. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n)!}$

54. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!}$

55. $a_n = \frac{1 + (-1)^n}{n}$

56. $a_n = \frac{1 + (-1)^n}{n^2}$

---

**Section 9.1**

$$\lim_{n \to \infty} \frac{1 + (0.2)^n}{n} = 0$$

If $n$ is odd, all terms are 2.

If $n$ is even, $\lim_{n \to \infty} \frac{2}{n} = 0$

By observation, the sequence converges to zero.

Even terms: $1, 2, 2, 2, 2, \ldots$

Odd terms: 0

$q_1 = 0$
$q_2 = \frac{1}{2}$
$q_3 = \frac{1}{2}$
$q_4 = 0$
$q_5 = \frac{1}{2}$
$q_6 = \frac{1}{2}$
$q_7 = 0$
$q_8 = \frac{1}{2}$
\[ Z \sim N^2 = -1, 1, -1, 1, -1, 1 \ldots \]

\[ UB = 1 \]
\[ LB = -1 \]
Not Monotonic

\[ \lim_{n \to \infty} (-1)^n \text{ Does Not Exist} \]

\[ \left\{ \frac{n^2}{2^n - 1} \right\} = \frac{1}{1}, \frac{4}{3}, \frac{9}{7}, \frac{16}{15}, \frac{25}{31} \ldots \]

\[ \lim_{x \to 0} \frac{x^2}{2^x - 1} = 8 \quad \text{DIV} \]

\[ = \frac{2x}{(\ln 2)2^x} = 8 \quad \text{L'Hopital} \]

\[ \lim_{n \to \infty} \left( \frac{2}{2^n + 2^x} \right) = 0 \quad \text{L'Hopital} \]
9.2 Series and Convergence

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the nth-Term Test for Divergence of an infinite series.

Infinite Series

One important application of infinite sequences is in representing “infinite summations.” Informally, if \( \{a_n\} \) is an infinite sequence, then

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

is an infinite series (or simply a series). The numbers \( a_1, a_2, a_3, \ldots \) are the terms of the series. For some series it is convenient to begin the index at \( n = 0 \) (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply \( \sum a_n \). In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following sequence of partial sums.

\[
S_1 = a_1
\]
\[
S_2 = a_1 + a_2
\]
\[
S_3 = a_1 + a_2 + a_3
\]
\[
\vdots
\]
\[
S_n = a_1 + a_2 + a_3 + \cdots + a_n
\]

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

<table>
<thead>
<tr>
<th>DEFINITIONS OF CONVERGENT AND DIVERGENT SERIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>For the infinite series ( \sum_{n=1}^{\infty} a_n ), the \textit{n}th partial sum is given by</td>
</tr>
</tbody>
</table>
| \[
S_n = a_1 + a_2 + \cdots + a_n
\]  |
| If the sequence of partial sums \( \{S_n\} \) converges to \( S \), then the series \( \sum_{n=1}^{\infty} a_n \) converges. The limit \( S \) is called the \textit{sum} of the series. |
| \[
S = a_1 + a_2 + \cdots + a_n + \cdots \quad S = \sum_{n=1}^{\infty} a_n
\]  |
| If \( \{S_n\} \) diverges, then the series diverges. |
Try It 1

Determine if the series
\[ \sum_{n=1}^{\infty} \frac{2}{3^n} \]
converges or diverges.

\[ S = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots \]

To one

\[ S_1 = \frac{2}{3} \]

\[ S_2 = \frac{2}{3} + \frac{2}{9} = \frac{6}{9} + \frac{2}{9} = \frac{8}{9} \]

\[ S_3 = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} = \frac{18}{27} + \frac{6}{27} + \frac{2}{27} = \frac{26}{27} \]

\[ S_4 = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} = \frac{54}{81} + \frac{18}{81} + \frac{6}{81} + \frac{2}{81} = \frac{80}{81} \]

Formule For

\[ S_n = \frac{\frac{2}{3} - 1}{\frac{3^n}{\text{sum of $n$ Terms of The Sereis}}} \]

\[ \lim_{n \to \infty} \frac{3^n - 1}{3^n} = \frac{8}{8} = 1 \] (END)

L'Hôpital's

\[ \lim_{n \to \infty} \frac{(\ln 3) 3^n}{(\ln 3) 3^n} = 1 \] Sum of The Terms Of The Series Converge To one!
b. The $n$th partial sum of the series
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots
\]
is given by
\[
S_n = 1 - \frac{1}{n+1}
\]
Because the limit of $S_n$ is 1, the series converges and its sum is 1.

The series in Example 1(b) is a **telescoping series** of the form
\[
(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots.
\]

Note that $b_2$ is canceled by the second term, $b_3$ is canceled by the third term, and so on. Because the $n$th partial sum of this series is
\[
S_n = b_1 - b_{n+1}
\]
it follows that a telescoping series will converge if and only if $b_n$ approaches a finite number as $n \to \infty$. Moreover, if the series converges, its sum is
\[
S = b_1 - \lim_{n \to \infty} b_{n+1}.
\]
Try It 2

Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \).

\[
\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots
\]

Telescope

\[
\frac{1}{n} - \frac{1}{n+1}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1
\]

Sum of Series
**Theorem 9.6 Convergence of a Geometric Series**

A geometric series with ratio $r$ diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$
\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.
$$

**Proof.** It is easy to see that the series diverges if $r = \pm 1$. If $r \neq \pm 1$, then $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplication by $r$ yields

$$
rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.
$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1 - r) = a(1 - r^n)$, and the $n$th partial sum is

$$
S_n = \frac{a}{1-r} (1 - r^n).
$$

If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{a}{1-r} (1 - r^n)\right] = \frac{a}{1-r} \lim_{n \to \infty} (1 - r^n) = \frac{a}{1-r}
$$

which means that the series converges and its sum is $a/(1 - r)$. It is left to you to show that the series diverges if $|r| > 1$. \hfill \blacksquare
Try It 3

Find the sum of the geometric series \( \sum_{n=0}^{\infty} 2 \left( \frac{2}{3} \right)^n \).

\[ 2 + \frac{4}{3} + \frac{16}{27} + \cdots \]

- \( a = 2 \)
- \( r = \frac{2}{3} \)

Condition:
- \( |r| < 1 \)

Sum:
\[ \frac{a}{1-r} = \frac{2}{1-\frac{2}{3}} \]
\[ = \frac{3}{1} \cdot \frac{2}{1} \]
\[ = 6 \]

\[ \text{Sum} = 6 \]
Try It 4

Use a geometric series to express 0.23 as the ratio of two integers.

\[ \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \frac{23}{100000000} + \cdots \]

\[ \sum_{n=0}^{\infty} \left( \frac{23}{100} \right) \left( \frac{1}{100} \right)^n \]

\[ \text{Sum} = \frac{9}{1 - r} = \frac{\frac{23}{100}}{1 - \frac{1}{100}} = \frac{23}{99} \frac{100}{99} \]

\[ \text{N} = 0.23 \]

\[ \frac{16000N}{100} = 342.23 \]

\[ N = 34.23 \]

\[ \frac{10000N}{9900} = 33.89 \]

\[ N = 33.89 \]
The following properties are direct consequences of the corresponding properties of limits of sequences.

**THEOREM 9.7 PROPERTIES OF INFINITE SERIES**

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let $A$, $B$, and $c$ be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$
**nth-Term Test for Divergence**

The following theorem states that if a series converges, the limit of its nth term must be 0.

**THEOREM 9.8 LIMIT OF THE nth TERM OF A CONVERGENT SERIES**

If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0. \)

**Proof** Assume that

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = L.
\]
Then, because \( S_n = S_{n-1} + a_n \) and

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} = L
\]

it follows that

\[
L = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (S_{n-1} + a_n)
= \lim_{n \to \infty} S_{n-1} + \lim_{n \to \infty} a_n
= L + \lim_{n \to \infty} a_n
\]

which implies that \( \{a_n\} \) converges to 0.

The contrapositive of Theorem 9.8 provides a useful test for divergence. This **nth-Term Test for Divergence** states that if the limit of the nth term of a series does not converge to 0, the series must diverge.

**THEOREM 9.9 nth-TERM TEST FOR DIVERGENCE**

If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.
Try It 5

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$.

Would Fail the nth Term Test

This Series Diverges

By nth Term Test

$\sum_{n=1}^{\infty} \frac{1}{2^n}$

$\lim_{n \to \infty} \frac{1}{2^n} = 0$

On Wednesday we can show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
Try It 6

A ball is dropped from a height of 14 feet and begins bouncing, as shown in the figure. The height of each bounce is nine-tenths the height of the previous bounce. Find the total vertical distance traveled by the ball.

\[ D_1 = 14 \]

\[ D_2 = \frac{9}{10}(14) + \frac{9}{10}(14) = 2 \cdot 14 \left(\frac{9}{10}\right) \]

\[ D_3 = \left(\frac{9}{10}\right)^2(14) + \left(\frac{9}{10}\right)^2(14) = 2 \cdot 14 \left(\frac{9}{10}\right)^2 \]

\[ D_4 = 2 \cdot 14 \left(\frac{9}{10}\right)^3 \]

\[ D_5 = 2 \cdot 14 \left(\frac{9}{10}\right)^4 \]

\[
14 + 2 \cdot 14 \left(\frac{9}{10}\right) + 2 \cdot 14 \left(\frac{9}{10}\right)^2 + 2 \cdot 14 \left(\frac{9}{10}\right)^3 + 2 \cdot 14 \left(\frac{9}{10}\right)^4 + \ldots
\]

\[
14 + \sum_{n=0}^{\infty} 2 \cdot 14 \left(\frac{9}{10}\right)^n (n+1)
\]

**Sum of Infinite Geometric Series**

\[
\frac{2 \cdot 14 \left(\frac{9}{10}\right)}{1 - \frac{9}{10}} = \frac{9}{1 - r}
\]

\[
\frac{2 \cdot 14 \left(\frac{9}{10}\right)}{1 - \frac{1}{10}} = \frac{2 \cdot 14 \cdot \frac{9}{10}}{1 - \frac{1}{10}} = 252
\]

\[
14 + 252 = 266 \text{ feet}
\]
9.2 Exercises

In Exercises 1–6, find the sequence of partial sums $S_1, S_2, S_3, S_4,$ and $S_n.$

1. $1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{10} + \frac{1}{20} + \cdots$
2. $\frac{1}{2} \cdot 3 + \frac{2}{3} \cdot 4 + \frac{3}{4} \cdot 5 + \frac{4}{5} \cdot 6 + \frac{5}{6} \cdot 7 + \cdots$
3. $3 - \frac{3}{2} + \frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \cdots$
4. $\frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$
5. $\sum_{n=1}^{\infty} \frac{3}{2^n}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

In Exercises 7 and 8, determine whether $\{a_n\}$ and $\sum a_n$ are convergent.

7. $a_n = \frac{n+1}{n}$
8. $a_n = 3\left(\frac{1}{5}\right)^n$

In Exercises 9–18, verify that the infinite series diverges.

9. $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$
10. $\sum_{n=0}^{\infty} \left(\frac{11}{10}\right)^n$
11. $\sum_{n=0}^{\infty} 1000(1.055)^n$
12. $\sum_{n=0}^{\infty} 2(-1.03)^n$
13. $\sum_{n=0}^{\infty} \frac{n}{n+1}$
14. $\sum_{n=0}^{\infty} \frac{n}{2n+3}$
15. $\sum_{n=0}^{\infty} \frac{n^2}{n^2+1}$
16. $\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$
17. $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$
18. $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

In Exercises 19–24, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Use the graph to estimate the sum of the series. Confirm your answer analytically.

(a) $\sum_{n=1}^{\infty} \frac{1}{n}$
(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(c) $\sum_{n=1}^{\infty} \frac{1}{n^3}$
(d) $\sum_{n=1}^{\infty} \frac{1}{n^4}$
(e) $\sum_{n=1}^{\infty} \frac{1}{n^5}$
(f) $\sum_{n=1}^{\infty} \frac{1}{n^6}$

In Exercises 25–30, verify that the infinite series converges.

25. $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$
26. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$
27. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$
28. $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$
29. $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ (Use partial fractions.)
30. $\sum_{n=0}^{\infty} \frac{1}{n(n+2)}$ (Use partial fractions.)

Numerical, Graphical, and Analytic Analysis. In Exercises 31–36, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum $S_n$ and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

31. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$
32. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$
33. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$
34. $\sum_{n=1}^{\infty} 3(0.85)^{n-1}$
35. $\sum_{n=1}^{\infty} 10(0.25)^{n-1}$
36. $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}$

In Exercises 37–52, find the sum of the convergent series.

37. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$
38. $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$
39. \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \)
40. \( \sum_{n=1}^{\infty} \left( \frac{6}{7} \right)^n \)
41. \( \sum_{n=1}^{\infty} \frac{1}{n^2 - 1} \)
42. \( \sum_{n=1}^{\infty} \frac{4}{n(n+2)} \)
43. \( \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \)
44. \( \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} \)
45. \( 1 + 0.1 + 0.01 + 0.001 + \cdots \)
46. \( 8 + 6 + \frac{4}{2} + \frac{2}{2} + \cdots \)
47. \( 3 - 1 + \frac{1}{2} - \frac{1}{3} + \cdots \)
48. \( 4 - 2 + 1 - \frac{1}{2} + \cdots \)
49. \( \sum_{n=1}^{\infty} \left( \frac{1}{2^n} - \frac{1}{3^n} \right) \)
50. \( \sum_{n=1}^{\infty} \left[ (0.7)^n + (0.9)^n \right] \)
51. \( \sum_{n=1}^{\infty} (\sin 1)^n \)
52. \( \sum_{n=1}^{\infty} \frac{1}{9n^3 + 3n - 2} \)

In Exercises 53–58, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.
53. \( 0.3 \overline{7} \)
54. \( 0.5 \overline{3} \)
55. \( 0.8 \overline{7} \)
56. \( 0.0 \overline{8} \)
57. \( 0.07 \overline{5} \)
58. \( 0.2 \overline{7} \overline{5} \)

In Exercises 59–76, determine the convergence or divergence of the series.
59. \( \sum_{n=1}^{\infty} (1.075)^n \)
60. \( \sum_{n=1}^{\infty} \frac{3^n}{1000} \)
61. \( \sum_{n=1}^{\infty} \frac{n + 10}{10n + 1} \)
62. \( \sum_{n=1}^{\infty} \frac{4n + 1}{3n - 1} \)
63. \( \sum_{n=1}^{\infty} \frac{1}{n - n + 2} \)
64. \( \sum_{n=1}^{\infty} \frac{1}{n + 1 - n + 2} \)
65. \( \sum_{n=1}^{\infty} \frac{1}{n(n + 3)} \)
66. \( \sum_{n=1}^{\infty} \frac{1}{2n(n + 1)} \)
67. \( \sum_{n=1}^{\infty} \frac{3n - 1}{2n + 1} \)
68. \( \sum_{n=1}^{\infty} \frac{3^n}{2^n} \)
69. \( \sum_{n=1}^{\infty} \frac{4^n}{2^n} \)
70. \( \sum_{n=1}^{\infty} \frac{3^n}{5^n} \)
71. \( \sum_{n=1}^{\infty} \frac{n}{n!} \)
72. \( \sum_{n=1}^{\infty} \frac{1}{n!} \)
73. \( \sum_{n=1}^{\infty} \left( \frac{1 + \frac{1}{n}}{n} \right) \)
74. \( \sum_{n=1}^{\infty} e^{-n} \)
75. \( \sum_{n=1}^{\infty} \arctan n \)
76. \( \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} \)

**WRITING ABOUT CONCEPTS**

**77.** State the definitions of convergent and divergent series.

**78.** Describe the difference between \( \lim_{n \to \infty} a_n = 5 \) and \( \sum_{n=1}^{\infty} a_n = 5 \).

**79.** Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

**80.** State the n-th Term Test for Divergence.

**WRITING ABOUT CONCEPTS (continued)**

**81.** Explain any differences among the following series.
(a) \( \sum_{n=1}^{\infty} a_n \)  (b) \( \sum_{n=1}^{\infty} a_n \)  (c) \( \sum_{n=1}^{\infty} a_n \)

**82.** (a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.
(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

In Exercises 83–90, find all values of \( x \) for which the series converges. For these values of \( x \), write the sum of the series as a function of \( x \).
83. \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)
84. \( \sum_{n=1}^{\infty} \frac{3^n x^n}{n} \)
85. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)
86. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n} \)
87. \( \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \)
88. \( \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \)
89. \( \sum_{n=1}^{\infty} \frac{1}{n} \)
90. \( \sum_{n=1}^{\infty} \frac{e^n}{n!} \)

In Exercises 91 and 92, find the value of \( c \) for which the series equals the indicated sum.
91. \( \sum_{n=1}^{\infty} (1 + c)^{-n} = 2 \)
92. \( \sum_{n=1}^{\infty} e^n = 5 \)

**93. Think About It**

Consider the formula
\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots
\]
Given \( x = -1 \) and \( x = 2 \), can you conclude that either of the following statements is true? Explain your reasoning.
(a) \( \frac{1}{2} = 1 + 1 + 1 + 1 + \cdots \)
(b) \( -1 = 1 + 2 + 4 + 8 + \cdots \)

**CAPSTONE**

**94. Think About It**

Are the following statements true? Why or why not?
(a) Because \( \frac{1}{n} \) approaches 0 as \( n \) approaches \( \infty \), \( \sum_{n=1}^{\infty} \frac{1}{n} \) converges.
(b) Because \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \), the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) converges.

In Exercises 95 and 96, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums \( S_n \) and \( S_{10} \). What do you notice?
95. \( 1 + x + x^2 + x^3 + \cdots \)
96. \( 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots \)
In Exercises 97 and 98, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.

97. \(f(x) = \begin{cases} \frac{1}{2} \cdot 0.5^x & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}\)

98. \(f(x) = \begin{cases} \frac{1}{2} \cdot 0.8^x & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}\)

Writing In Exercises 99 and 100, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.

99. \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n\)

100. \(\sum_{n=1}^{\infty} \frac{1}{2^n}, \sum_{n=1}^{\infty} (0.01)^n\)

101. Marketing An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, \([8000 + 0.95(8000)]\) units will be in use after 2 years, and so on. How many units will be in use after 5 years?

102. Depreciation A company buys a machine for $475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after \(n\) years. What is its value after 5 years?

103. Multiplier Effect The total annual spending by tourists in a resort city is $200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the $200 million and find the sum of the series.

104. Multiplier Effect Repeat Exercise 103 if the percent of the revenue that is spent in again decreases to 60%.

105. Distance A ball is dropped from a height of 16 feet. Each time it drops \(h\) feet, it rebounds 0.81\(h\) feet. Find the total distance traveled by the ball.

106. Time The ball in Exercise 105 takes the following times for each fall.

\[s_i = -16t^2 + 16, \quad s_i = 0 \text{ if } t = 1\]

\[s_2 = -16t^2 + 16(0.81), \quad s_2 = 0 \text{ if } t = 0.9\]

\[s_3 = -16t^2 + 16(0.81)^2, \quad s_3 = 0 \text{ if } t = (0.9)^2\]

\[s_4 = -16t^2 + 16(0.81)^3, \quad s_4 = 0 \text{ if } t = (0.9)^3\]

\[\vdots\]

\[s_n = -16t^2 + 16(0.81)^{n-1}, \quad s_n = 0 \text{ if } t = (0.9)^{n-1}\]

Beginning with \(s_2\), the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by \(t = 1 + 2 \sum_{n=1}^{\infty} (0.9)^n\). Find this total time.

Probability In Exercises 107 and 108, the random variable \(n\) represents the number of units of a product sold per day in a store. The probability distribution of \(n\) is given by \(P(n)\). Find the probability that two units are sold in a given day [\(P(2)\)] and show that \(P(0) + P(1) + P(2) + P(3) + \cdots = 1\).

107. \(P(n) = \frac{1}{\binom{n}{2}}\)

108. \(P(n) = \frac{1}{\binom{n}{3}}\)

109. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the \(n\)th toss is given by \(P(n) = \left(\frac{1}{2}\right)^n\), where \(n \geq 1\).

(a) Show that \(\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1\).

(b) The expected number of tosses required until the first head occurs in the experiment is given by \(\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n\). Is this series geometric?

CAS (c) Use a computer algebra system to find the sum in part (b).

110. Probability In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

111. Area The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) if this process is continued five more times and (b) if this pattern of shading is continued infinitely.

112. Length A right triangle XYZ is shown above where \(|XY| = z\) and \(\angle X = \theta\). Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments \(|y_1| + |x_0 y_1| + |x_1 y_2| + \cdots\) in terms of \(z\) and \(\theta\).

(b) If \(z = 1\) and \(\theta = \pi/6\), find the total length of the perpendicular line segments.

In Exercises 113–116, use the formula for the \(n\)th partial sum of a geometric series

\[\sum_{n=1}^{\infty} ar^n = \frac{a(1 - r^n)}{1 - r}\].

113. Present Value The winner of a $2,000,000 sweepstakes will be paid $100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is \(\sum_{n=1}^{20} 100,000 \frac{1}{(1.06)^n}\). Compute the present value and interpret its meaning.
114. **Sphereflake** The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius \( \frac{1}{2} \) are attached. To each of these, nine spheres of radius \( \frac{1}{4} \) are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.

115. **Salary** You go to work at a company that pays \$0.01 for the first day, \$0.02 for the second day, \$0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?\(^9\)

116. **Annuities** When an employee receives a paycheck at the end of each month, \( P \) dollars is invested in a retirement account. These deposits are made each month for \( t \) years and the account earns interest at the annual percentage rate \( r \). If the interest is compounded monthly, the amount \( A \) in the account at the end of \( t \) years is

\[
A = P \left( 1 + \frac{r}{12} \right)^{12t} + \cdots + P \left( 1 + \frac{r}{12} \right)^{12(t-1)}
\]

If the interest is compounded continuously, the amount \( A \) in the account after \( t \) years is

\[
A = Pe^{rt}
\]

Verify the formulas for the sums given above.

**Annuities** In Exercises 117–120, consider making monthly deposits of \( P \) dollars in a savings account at an annual interest rate \( r \). Use the results of Exercise 116 to find the balance \( A \) after \( t \) years if the interest is compounded (a) monthly and (b) continuously.

117. \( P = \$45, \ r = 3\%, \ t = 20 \) years

118. \( P = \$75, \ r = 5.5\%, \ t = 25 \) years

119. \( P = \$100, \ r = 4\%, \ t = 35 \) years

120. \( P = \$30, \ r = 6\%, \ t = 50 \) years

121. **Salary** You accept a job that pays a salary of \$50,000 for the first year. During the next 39 years you receive a 4% raise each year. What would be your total compensation over the 40-year period?

122. **Salary** Repeat Exercise 121 if the raise you receive each year is 4.5%. Compare the results.

**True or False?** In Exercises 123–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

123. If \( \lim_{n \to \infty} a_n = 0 \) then \( \sum_{n=1}^{\infty} a_n \) converges.

124. If \( \sum_{n=1}^{\infty} a_n = L \) then \( \sum_{n=1}^{\infty} a_n = L + a_0 \).

125. If \( |r| < 1 \), then \( \sum_{n=1}^{\infty} ar^n = \frac{a}{1 - r} \).

126. The series \( \sum_{n=1}^{\infty} \frac{n}{10000(n+1)} \) diverges.

127. \( 0.75 = 0.749999\ldots \).

128. Every decimal with a repeating pattern of digits is a rational number.

129. Show that the series \( \sum_{n=1}^{\infty} a_n \) can be written in the telescoping form

\[
\sum_{n=1}^{\infty} \left[(c - S_{n-1}) - (c - S_n)\right]
\]

where \( S_0 = 0 \) and \( S_n \) is the \( n \)th partial sum.

130. Let \( \Sigma a_n \) be a convergent series, and let

\[
R_N = a_{N+1} + a_{N+2} + \cdots
\]

be the remainder of the series after the first \( N \) terms. Prove that \( \lim_{N \to \infty} R_N = 0 \).

131. Find two divergent series \( \Sigma a_n \) and \( \Sigma b_n \) such that \( \Sigma (a_n + b_n) \) converges.

132. Given two infinite series \( \Sigma a_n \) and \( \Sigma b_n \) such that \( \Sigma a_n \) converges and \( \Sigma b_n \) diverges, prove that \( \Sigma (a_n + b_n) \) diverges.

133. Suppose that \( \Sigma a_n \) diverges and \( c \) is a nonzero constant. Prove that \( \Sigma ca_n \) diverges.

134. If \( \sum a_n \) converges where \( a_n \) is nonzero, show that \( \sum \frac{1}{a_n} \) diverges.

135. The Fibonacci sequence is defined recursively by \( a_{n+2} = a_{n+1} + a_n \), where \( a_1 = 1 \) and \( a_2 = 1 \).

(a) Show that \( \frac{1}{a_{n+1}a_{n+2}} = \frac{1}{a_{n+1}} - \frac{1}{a_{n+2}} \).

(b) Show that \( \sum \frac{1}{a_{n+1}a_{n+2}} = 1 \).

136. Find the value of \( \alpha \) for which the infinite series

\[
1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + 5\alpha^4 + \cdots
\]

converges. What is the sum when the series converges?

137. Prove that \( \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \frac{1}{r - 1} \) for \( |r| > 1 \).
138. Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \).

*Hint:* Find the constants \( A \), \( B \), and \( C \) such that
\[
\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}.
\]

139. (a) The integrand of each definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.
\[
\int_0^1 (1-x) \, dx \quad \int_0^1 (x-x^2) \, dx \quad \int_0^1 (x^2-x^3) \, dx
\]
(b) Find the area of each region in part (a).
(c) Let \( a_n = \int_0^1 (x^{n-1} - x^n) \, dx \). Evaluate \( a_n \) and \( \sum_{n=1}^{\infty} a_n \). What do you observe?

140. **Writing** The figure below represents an informal way of showing that \( \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 \). Explain how the figure implies this conclusion.

![Figure](image)

**SECTION PROJECT**

**Cantor's Disappearing Table**

The following procedure shows how to make a table disappear by removing only half of the table:

(a) Original table has a length of \( L \).

![Table](image)

(b) Remove \( \frac{1}{4} \) of the table centered at the midpoint. Each remaining piece has a length that is less than \( \frac{L}{2} \).

![Table](image)

(c) Remove \( \frac{1}{4} \) of the table by taking sections of length \( \frac{L}{2} \) from the centers of each of the two remaining pieces. Now, you have removed \( \frac{1}{4} + \frac{1}{4} \) of the table. Each remaining piece has a length that is less than \( \frac{L}{4} \).

![Table](image)

(d) Remove \( \frac{1}{16} \) of the table by taking sections of length \( \frac{L}{4} \) from the centers of each of the four remaining pieces. Now, you have removed \( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \) of the table. Each remaining piece has a length that is less than \( \frac{L}{16} \).

![Table](image)

Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

**FOR FURTHER INFORMATION** Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

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**FOR FURTHER INFORMATION** For more on this exercise, see the article "Convergence with Pictures" by P.J. Rippon in *American Mathematical Monthly*.

**PUTNAM EXAM CHALLENGE**

142. Write \( \sum_{k=1}^{\infty} \frac{6^k}{(3^k+1)(3^k-2^k)} \) as a rational number.

143. Let \( f(n) \) be the sum of the first \( n \) terms of the sequence \( 0, 1, 1, 2, 2, 3, 3, 4, \ldots \), where the \( n \)th term is given by
\[
a_n = \begin{cases} 
n/2, & \text{if } n \text{ is even} \\
(n-1)/2, & \text{if } n \text{ is odd.} 
\end{cases}
\]

Show that if \( x \) and \( y \) are positive integers and \( x > y \) then
\[
x^y - f(x+y) = f(x-y).
\]

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.
(31) \[ \sum_{n=1}^{\infty} \frac{6}{n( n+3 )} = \frac{A}{n} + \frac{B}{n+3} \]
\[ = \frac{2}{n} - \frac{3}{n+3} \]

Write the first four terms:
\[ (2 - \frac{2}{4}) + (\frac{3}{2} - \frac{3}{5}) + (\frac{3}{3} - \frac{3}{6}) + (\frac{3}{4} - \frac{3}{7}) + (\frac{3}{5} - \frac{3}{8}) \times (\frac{2}{6} - \frac{3}{9}) \]

Write the last four terms:
\[ (\frac{3}{n+3} - \frac{3}{n+1}) + (\frac{3}{n+2} - \frac{3}{n+1}) + (\frac{3}{n+1} - \frac{3}{n}) + (\frac{3}{n} - \frac{3}{n+1}) \]

Limit \[ \lim_{n \to \infty} \left[ \sum_{n=1}^{\infty} \left( \frac{6}{n( n+3 )} \right) \right] = \lim_{n \to \infty} \left[ \sum_{n=1}^{\infty} \left( \frac{2}{n} - \frac{3}{n+3} \right) \right] \]
\[ = 2 + \frac{3}{2} + \frac{3}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3} \]
\[ = \frac{4}{3} \]

\[ \frac{4}{3} \]