Sectin 9.8

\[ \lim_{n \to \infty} \frac{\sum_{n=0}^{\infty} \frac{(2n)!}{(2n)!} \left( \frac{x}{3} \right)^n}{(2n)!} = \lim_{n \to \infty} \left| \frac{(2n+2)(2n+1)(2n)! \left( \frac{x}{3} \right)}{(2n)!} \right| \]

Converges only for center \( x = 0 \)
\( f(x) = \lim_{n \to \infty} \left( \frac{x^n}{n+1} \right) = \begin{cases} 0 & 0 < x < 1 \\ \infty & x > 1 \end{cases} \)

\( R \in c = 1 \)

\( \sum_{n=0}^{\infty} (-1)^n C_n \sin x \)

\( f'(x) = \sum_{n=0}^{\infty} (-1)^n (n+1)(x^{n+1}) = \sum_{n=0}^{\infty} (x-1)^n \)

\( S f(x) dx: \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{n+2}}{(n+1) (n+2)} \)

\( X = 0 \)

\( \sum_{n=1}^{\infty} \frac{(-1)^n (1)^{n+2}}{n^3 + 3n^2 + 2} \)

\( X = 2 \)

\( \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n^3 + 3n^2 + 2} \)

Prove convergence

By Alternating Series Test

\([0, 2]\)
representation of Functions by Power Series

- Find a geometric power series that represents a function.
- Construct a power series using series operations.

Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a given function.

Consider the function given by $f(x) = 1/(1 - x)$. The form of $f$ closely resembles the sum of a geometric series:

$$
\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad |r| < 1.
$$

In other words, if you let $a = 1$ and $r = x$, a power series representation for $1/(1 - x)$, centered at 0, is

$$
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.
$$

Of course, this series represents $f(x) = 1/(1 - x)$ only on the interval $(-1, 1)$, whereas $f$ is defined for all $x \neq 1$, as shown in Figure 9.22. To represent $f$ in another interval, you must develop a different series. For instance, to obtain the power series centered at $-1$, you could write

$$
\frac{1}{1 - x} = \frac{1}{2 - (x + 1)} = \frac{1/2}{1 - [(x + 1)/2]} = \frac{a}{1 - r}
$$

which implies that $a = \frac{1}{2}$ and $r = (x + 1)/2$. So, for $|x + 1| < 2$, you have

$$
\frac{1}{1 - x} = \sum_{n=0}^{\infty} \frac{(x + 1)}{2^n} = \frac{1}{2}
$$

$$
\frac{1}{1 - x} = \sum_{n=0}^{\infty} \left[ 1 + \frac{(x + 1)}{2} + \frac{(x + 1)^2}{4} + \frac{(x + 1)^3}{8} + \cdots \right], \quad |x + 1| < 2
$$

which converges on the interval $(-3, 1)$.
Try It 1

\[ f(x) = \frac{\sum_{n=0}^{\infty} \frac{3}{4} (\frac{x}{4})^n}{1 - \frac{x}{4}} = \frac{\sum_{n=0}^{\infty} \frac{3}{4} \cdot \frac{x^n}{4^n}}{1 - \frac{x}{4}} \]

Find a power series for \( f(x) = \frac{3}{4 - x} \), centered at 0.

1) Center It
2) Compute \( \frac{1}{1-r} \)
3) Get A One
4) Make Sure You Have 1-r

\[ f(x) = \frac{3}{4-x} = \sum_{n=0}^{\infty} \frac{3}{4} \cdot \frac{x^n}{4^n} \]

\[ r = \frac{x}{4} \]

\[ |\frac{x}{4}| < 1 \]

\[-1 < \frac{x}{4} < 1 \]
\[-4 < x < 4 \]

Interval of Convergence (I0C): (-4, 4)

Don't Need To Check The End pts. Since It Is A Geometric Seriess.
Try It 2

Find a power series for \( f(x) = \frac{3}{4 - x} \), centered at \(-2\).

\[
\frac{3}{2 + 4 - (x+2)} = \frac{3}{6 - (x+2)} = \frac{3}{6 - \frac{x+2}{6}}
\]

\[
\frac{3}{4-x} = \frac{\frac{1}{6}}{1 - \frac{x+2}{6}} = \frac{9}{1-x}
\]

Geometric Series

\[
\sum_{n=0}^{\infty} a n = \sum_{n=0}^{\infty} \left( \frac{x}{6} \right) \left( \frac{x+2}{6} \right)^n
\]

Geometric Series

\[ |r| < 1 \]

\[ -1 < \frac{x+2}{6} < 1 \]

\[ -6 < x+2 < 6 \]

\[ -2 < x < 4 \]

The Endpts. Don't Work In A Geometric Series

IOC

\((-8,4)\)
Example \( g(x) = \frac{1}{2x-5} \)

Centered \( A+x=3 \)

\[ x-(3) = x+3 \]

\[ \frac{1}{2x-5} = \frac{1}{-5 + 2(x+3)} \]

1) Center in
2) Compose for centering
3) Get A + r
   Where, the
   -1 \( r \) is.

\[ \frac{\frac{1}{11}}{-\frac{11}{11} + \frac{2}{11}(x+3)} = \frac{\frac{1}{11}}{-1 + \frac{2}{11}(x+3)} \]

\[ \frac{\frac{1}{11}}{-1 + \frac{2}{11}(x+3)} + \frac{2}{11}(x+3) \]

\[ = \frac{\frac{1}{11}}{1 + \frac{2}{11}(x+3)} \]

\[ = \frac{\frac{1}{11}}{1 - \frac{2}{11}(x+3)} = \frac{q}{1-q} \]

\[ q = \frac{\frac{1}{11}}{1} \]

\[ r = \frac{2}{11}(x+3) \]

\[ \sum_{n=0}^{\infty} (\frac{-\frac{2}{11}}{2(x+3)})^n = f(x) \]

\[ \sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}} = f(x) \]

\[ \left| \frac{\frac{2}{11}(x+3)}{1} \right| < 1 \]

-1 < \( \frac{2(x+3)}{1} \) < 1

-11 < 2x+6 < 11

-11 < 2x < 5

-\( \frac{11}{2} < x < \frac{5}{2} \)

 IOC

\( [-\frac{11}{2}, \frac{5}{2}] \)

\[ \frac{1}{2x-5} \text{ converges to} \]

\[ \frac{1}{2x-5} \text{ between} \]

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Operations with Power Series

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. (For simplicity, the following properties are stated for a series centered at 0.)

<table>
<thead>
<tr>
<th>OPERATIONS WITH POWER SERIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( f(x) = \sum a_n x^n ) and ( g(x) = \sum b_n x^n ).</td>
</tr>
<tr>
<td>1. ( f(kx) = \sum a_n k^n x^n )</td>
</tr>
<tr>
<td>2. ( f(x^n) = \sum a_n x^{nk} )</td>
</tr>
<tr>
<td>3. ( f(x) \pm g(x) = \sum (a_n \pm b_n) x^n )</td>
</tr>
</tbody>
</table>

The operations described above can change the interval of convergence for the resulting series. For example, in the following addition, the interval of convergence for the sum is the intersection of the intervals of convergence of the two original series.

\[
\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right)x^n
\]

Intersection of The Intervals of The Original Series

Sum of The Two Series
Try It 3

Find a power series, centered at 0, for \( f(x) = \frac{2x}{x^2 - 1} = \frac{2x}{(x+1)(x-1)} \).

\[ \frac{2x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \]

\[ 2x = A(x-1) + B(x+1) \]

<table>
<thead>
<tr>
<th>( x = 1 )</th>
<th>( x = -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 = 2B</td>
<td>-2 = -2A</td>
</tr>
<tr>
<td>B = 1</td>
<td>1 = A</td>
</tr>
</tbody>
</table>

\[ \frac{1}{1+4x} = \frac{1}{1-(-4x)} = \frac{q}{1-r} \]

\( q = 1 \)
\( r = -4x \)

\[ \sum_{n=0}^{\infty} (-1)^n x^n \]

\[ \frac{1}{1-x} = \frac{1}{1-(-4x)} = \frac{a}{1-r} \]

\( a = -1 \)
\( r = 4x \)

\[ \sum_{n=0}^{\infty} (-1)^n x^n \]

\[ \sum_{n=0}^{\infty} (x^1)^n - \sum_{n=0}^{\infty} x^n \]

\[ \sum_{n=0}^{\infty} (-1)^n (x^1)^n \]

\[ -1 < x < 1 \]

\[ IOC = (-1, 1) \]
Try It 4

Find a power series for \( f(x) = \ln(x^3 + 1) \), centered at 0.

\[
\frac{d}{dx} \left[ \ln(x^3 + 1) \right] = \frac{3x^2}{x^3 + 1} = \frac{3x^2}{1 + x^{-3}}
\]

\[
g(x) = \frac{1}{x^3 + 1} = \sum_{n=0}^{\infty} (-1)^n x^{3n}
\]

\[
g_n = \frac{3^{n+1} \cdot 1}{x^3 + 1} = \frac{3^{n+1}}{x^3}
\]

\[
\ln (x^3 + 1) = \sum_{n=0}^{\infty} \frac{3x^{3n}}{n+1} + C
\]

\[
\ln (x^3 + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{x^{3n}} + C
\]

\[
\ln (x^3 + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{x^{3n}} + C
\]

Let \( x = 0 \). To find \( C \):

\[
\ln(1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{x^{3n}} + C
\]

\[
0 = C
\]

\[
\ln (x^3 + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{x^{3n}} = \frac{x^3}{3} - \frac{x^6}{6} + \frac{x^9}{9} - \frac{x^{12}}{12} + \ldots
\]

To see when the series converges:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3^{3n+1} \cdot x^3}{3^{3n} \cdot x^3} \right| = \lim_{n \to \infty} \left| \frac{3}{1} \right| = 3
\]

\[
|x^3| \leq 1
\]

Check bounds:

\[
|x^3| < 1 \quad \Rightarrow \quad x < 1
\]

\[
|x^3| > 1 \quad \Rightarrow \quad x > 1
\]
EXAMPLE 5 Finding a Power Series by Integration

Find a power series for \( g(x) = \arctan x \), centered at 0.

**Solution** Because \( D_x[\arctan x] = 1/(1 + x^2) \), you can use the series

\[
f(x) = \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n.
\]

Interval of convergence: \((-1, 1)\)

Substituting \( x^2 \) for \( x \) produces

\[
f(x^2) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.
\]

Integrate

Finally, by integrating, you obtain

\[
\arctan x = \int \frac{1}{1 + x^2} \, dx + C
\]

\[
= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}
\]

Let \( x = 0 \), then \( C = 0 \).

Interval of convergence: \((-1, 1)\)

\[
= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.
\]

Homework

Run a Ratio Test
Check End pts
In Section 9.7, the fourth-degree Taylor polynomial for the natural logarithmic function

\[ \ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} \]

was used to approximate \( \ln(1.1) \).

\[ \ln(1.1) = (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \]

\[ = 0.0953083 \]

You now know from Example 4 that this polynomial represents the first four terms of the power series for \( \ln x \). Moreover, using the Alternating Series Remainder, you can determine that the error in this approximation is less than

\[ |R_4| \leq |a_5| \]
\[ = \frac{1}{5}(0.1)^5 \]
\[ = 0.000002. \]

During the seventeenth and eighteenth centuries, mathematical tables for logarithms and values of other transcendental functions were computed in this manner. Such numerical techniques are far from outdated, because it is precisely by such means that many modern calculating devices are programmed to evaluate transcendental functions.
Try It 5

Find a power series for $g(x) = \arctan(3x)$, centered at 0.
It can be shown that the power series developed for \( \arctan x \) in Example 5 also converges (to \( \arctan x \)) for \( x = \pm 1 \). For instance, when \( x = 1 \), you can write

\[
\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

\[
= \frac{\pi}{4}
\]

However, this series (developed by James Gregory in 1671) does not give us a practical way of approximating \( \pi \) because it converges so slowly that hundreds of terms would have to be used to obtain reasonable accuracy. Example 6 shows how to use two different arctangent series to obtain a very good approximation of \( \pi \) using only a few terms. This approximation was developed by John Machin in 1706.

**Try It 6**

Use the trigonometric identity

\[
2 \arctan \frac{2}{3} - \arctan \frac{7}{17} = \frac{\pi}{4}
\]

to approximate the number \( \pi \).

**Solution**

By using only six terms from each of the series for \( \arctan(2/3) \) and \( \arctan(7/17) \), you obtain

\[
4\left(2 \arctan \frac{2}{3} - \arctan \frac{7}{17}\right) \approx 3.1415926
\]

which agrees with the exact value of \( \pi \) with an error less than 0.0000001.
9.9 Exercises

In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

1. \( f(x) = \frac{1}{4 - x} \)
2. \( f(x) = \frac{1}{2 + x} \)
3. \( f(x) = \frac{3}{4 + x} \)
4. \( f(x) = \frac{2}{5 - x} \)

In Exercises 5–16, find a power series for the function, centered at \( c \), and determine the interval of convergence.

5. \( f(x) = \frac{1}{3 - x}, \quad c = 1 \)
6. \( f(x) = \frac{4}{5 - x}, \quad c = -3 \)
7. \( f(x) = \frac{1}{1 - 3x}, \quad c = 0 \)
8. \( h(x) = \frac{1}{1 - 5x}, \quad c = 0 \)
9. \( g(x) = \frac{5}{2x - 3}, \quad c = -3 \)
10. \( f(x) = \frac{3}{2x - 1}, \quad c = 2 \)
11. \( f(x) = \frac{2}{2x + 3}, \quad c = 0 \)
12. \( f(x) = \frac{4}{3x + 2}, \quad c = 3 \)
13. \( g(x) = \frac{4}{x^2 + 2x - 3}, \quad c = 0 \)
14. \( g(x) = \frac{3x - 8}{3x^2 + 5x - 2}, \quad c = 0 \)
15. \( f(x) = \frac{2}{1 - x^2}, \quad c = 0 \)
16. \( f(x) = \frac{5}{5 + x^2}, \quad c = 0 \)

In Exercises 17–26, use the power series

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n
\]

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

17. \( h(x) = \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x} \)
18. \( h(x) = \frac{x}{x^2 - 1} = \frac{1}{2(1 + x)} - \frac{1}{2(1 - x)} \)
19. \( f(x) = \frac{1}{(x + 1)^2} = \frac{d}{dx} \left[ \frac{1}{x + 1} \right] \)
20. \( f(x) = \frac{1}{(x + 1)^3} = \frac{d^2}{dx^2} \left[ \frac{1}{x + 1} \right] \)
21. \( f(x) = \ln(x + 1) = \int \frac{1}{x + 1} \, dx \)
22. \( f(x) = \ln(1 - x^2) = \int \frac{1}{1 + x} \, dx - \int \frac{1}{1 - x} \, dx \)
23. \( g(x) = \frac{1}{x^2 + 1} \)
24. \( f(x) = \ln(x^2 + 1) \)
25. \( h(x) = \frac{1}{4x^2 + 1} \)
26. \( f(x) = \arctan 2x \)

Graphical and Numerical Analysis

In Exercises 27 and 28, let

\[
S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n},
\]

Use a graphing utility to verify the inequality graphically. Then complete the table to confirm the inequality numerically.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n )</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_{n+1} )</td>
<td></td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

27. \( S_2 \leq \ln(x + 1) \leq S_3 \)
28. \( S_4 \leq \ln(x + 1) \leq S_5 \)

In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when \( x = 0.5 \), and (d) determine what the approximation represents and how good the approximation is.

29. \( \sum_{n=1}^{\infty} (-1)^{n+1} (x - 1)^n \)
30. \( \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \)

In Exercises 31–34, match the polynomial approximation of the function \( f(x) = \arctan x \) with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

31. \( g(x) = x \)
32. \( g(x) = x - x^3 \)
33. \( g(x) = x - x^3 + x^5 \)
34. \( g(x) = x - x^3 + x^5 - x^7 \)
In Exercises 35–38, use the series for \( f(x) = \arctan x \) to approximate the value, using \( R_n \leq 0.001 \).

35. \( \arctan \frac{1}{4} \) \hspace{1cm} 36. \( \int_0^{\pi/4} \arctan x^2 \, dx \)

37. \( \int_0^{\pi/4} \arctan x^2 \, dx \) \hspace{1cm} 38. \( \int_0^{\pi/4} x \arctan x \, dx \)

In Exercises 39–42, use the power series

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1. \]

Find the series representation of the function and determine its interval of convergence.

39. \( f(x) = \frac{1}{(1-x)^2} \)

40. \( f(x) = \frac{x}{(1-x)^2} \)

41. \( f(x) = \frac{1+x}{(1-x)^3} \)

42. \( f(x) = \frac{4(1+x)}{(1-x)^3} \)

43. **Probability** A fair coin is tossed repeatedly. The probability that the first head occurs on the \( n \)th toss is \( P(n) = \left( \frac{1}{2} \right)^n \). When this game is repeated many times, the average number of tosses required until the first head occurs is

\[ E(n) = \sum_{n=1}^{\infty} nP(n). \]

(This value is called the expected value of \( n \).) Use the results of Exercises 39–42 to find \( E(n) \). Is the answer what you expected? Why or why not?

44. Use the results of Exercises 39–42 to find the sum of each series.

(a) \( \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \) \hspace{1cm} (b) \( \frac{1}{10} \sum_{n=1}^{\infty} \left( \frac{9}{10} \right)^n \)

**Writing** In Exercises 45–48, explain how to use the geometric series

\[ g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \]

to find the series for the function. Do not find the series.

45. \( f(x) = \frac{1}{1+x} \)

46. \( f(x) = \frac{1}{1-x^2} \)

47. \( f(x) = \frac{5}{1+x} \)

48. \( f(x) = \ln(1-x) \)

49. Prove that \( \arctan x + \arctan y = \arctan \frac{x+y}{1-xy} \) for \( xy \neq 1 \). Provided the value of the left side of the equation is between \(-\pi/2\) and \( \pi/2 \).

50. Use the result of Exercise 49 to verify each identity.

(a) \( \arctan \frac{120}{119} = \arctan \frac{\pi}{4} \)

(b) \( 4 \arctan \frac{1}{5} = \arctan \frac{\pi}{4} \)

[Hint: Use Exercise 49 twice to find \( 4 \arctan \frac{1}{2} \). Then use part (a).]

In Exercises 51 and 52, verify the given equation and (b) use the equation and the series for the arctangent to approximate \( \pi \) to two-decimal-place accuracy.

51. \( \frac{1}{2} - \arctan \frac{1}{7} = \frac{\pi}{4} \) \hspace{1cm} 52. \( \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4} \)

In Exercises 53–58, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

53. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{1}{2n} \right) \)

54. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{1}{3n} \right) \)

55. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{2}{5n} \right) \)

56. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{1}{2n+1} \right) \)

57. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{1}{2n+1} \right) \)

58. \( \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{1}{3n+1} \right) \)

**WRITING ABOUT CONCEPTS**

59. **Convergence and Divergence** Use the results of Exercises 31–34 to make a geometric argument for why the series approximations of \( f(x) = \arctan x \) have only odd powers of \( x \).

60. Use the results of Exercises 31–34 to make a conjecture about the degrees of series approximations of \( f(x) = \arctan x \) that have relative extrema.

61. One of the series in Exercises 53–58 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

62. The radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n x^n \) is 3. What is the radius of convergence of the series \( \sum_{n=0}^{\infty} n a_n x^{n-1} \)? Explain.

63. The power series \( \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x+1| < 4 \). What can you conclude about the series \( \sum_{n=0}^{\infty} a_n x^{n+1} \)? Explain.

**CAPSTONE**

64. **Find the Error** Describe why the statement is incorrect.

\[ \sum_{n=0}^{\infty} a_n x^n + \sum_{n=5}^{\infty} c_n (5-x)^n = \sum_{n=0}^{\infty} \left[ \frac{1}{5} + \frac{1}{5} x^n \right] \]

In Exercises 65 and 66, find the sum of the series.

65. \( \sum_{n=0}^{\infty} (-1)^n \frac{(n)}{3^n(2n+1)} \)

66. \( \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}(2n+1)!}{(2n+1)!} \)

**Ramanujan and Pi** Use a graphing utility to show that

\[ \frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{(4n)!}{(n)!2^{8n+1}} \]

\[ \text{Hint: Use Exercise 49 twice to find } 4 \arctan \frac{1}{2}. \text{ Then use part (a).} \]

\[ \frac{\pi}{9801} = \sum_{n=0}^{\infty} \frac{(4n)!}{(n)!2^{3n}} \frac{(1103 + 26390n)}{39690} = \frac{1}{\pi} \]