Chapter 14.1 to 14.6 Formulas

**AREA OF A REGION IN THE PLANE**

1. If \( R \) is defined by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), where \( g_1 \) and \( g_2 \) are continuous on \([a, b] \), then the area of \( R \) is given by
   \[
   A = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} dy \, dx.
   \]
   Figure 14.2 (vertically simple)

2. If \( R \) is defined by \( c \leq y \leq d \) and \( h_1(y) \leq x \leq h_2(y) \), where \( h_1 \) and \( h_2 \) are continuous on \([c, d] \), then the area of \( R \) is given by
   \[
   A = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} dx \, dy.
   \]
   Figure 14.3 (horizontally simple)

**DEFINITION OF DOUBLE INTEGRAL**

If \( f \) is defined on a closed, bounded region \( R \) in the \( xy \)-plane, then the double integral of \( f \) over \( R \) is given by
\[
\iint_{R} f(x, y) \, dA = \lim_{\|A\| \to 0} \sum_{1}^{n} f(x_i, y_i) \Delta A_i
\]
provided the limit exists. If the limit exists, then \( f \) is integrable over \( R \).

**VOLUME OF A SOLID REGION**

If \( f \) is integrable over a plane region \( R \) and \( f(x, y) \geq 0 \) for all \((x, y)\) in \( R \), then the volume of the solid region that lies above \( R \) and below the graph of \( f \) is defined as
\[
V = \iiint f(x, y) \, dA.
\]

**THEOREM 14.2 FUBINI’S THEOREM**

Let \( f \) be continuous on a plane region \( R \).

1. If \( R \) is defined by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), where \( g_1 \) and \( g_2 \) are continuous on \([a, b] \), then
   \[
   \iint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
   \]

2. If \( R \) is defined by \( c \leq y \leq d \) and \( h_1(y) \leq x \leq h_2(y) \), where \( h_1 \) and \( h_2 \) are continuous on \([c, d] \), then
   \[
   \iint_{R} f(x, y) \, dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
   \]
DEFINITION OF THE AVERAGE VALUE OF A FUNCTION OVER A REGION

If \( f \) is integrable over the plane region \( R \), then the average value of \( f \) over \( R \) is

\[
\frac{1}{A} \int_R f(x, y) \, dA
\]

where \( A \) is the area of \( R \).

THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM

Let \( R \) be a plane region consisting of all points \( (x, y) = (r \cos \theta, r \sin \theta) \) satisfying the conditions \( 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \), where

\[
0 \leq (\beta - \alpha) \leq 2\pi.
\]

If \( g_1 \) and \( g_2 \) are continuous on \([\alpha, \beta]\) and \( f \) is continuous on \( R \), then

\[
\int_R f(x, y) \, dA = \int_0^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
\]

NOTE: If \( z = f(x, y) \) is nonnegative on \( R \), then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of \( f \) and the region \( R \). When using the integral in Theorem 14.3, be sure not to omit the extra factor of \( r \) in the integrand.

The region \( R \) is restricted to two basic types, \( r \)-simple regions and \( \theta \)-simple regions, as shown in Figure 14.29.

DEFINITION OF MASS OF A PLANAR LAMINA OF VARIABLE DENSITY

If \( \rho \) is a continuous density function on the lamina corresponding to a plane region \( R \), then the mass \( m \) of the lamina is given by

\[
m = \int_R \rho(x, y) \, dA.
\]

Variable density

MOMENTS AND CENTER OF MASS OF A VARIABLE DENSITY PLANAR LAMINA

Let \( \rho \) be a continuous density function on the planar lamina \( R \). The moments of mass with respect to the \( x \)- and \( y \)-axes are

\[
M_x = \int_R y \rho(x, y) \, dA \quad \text{and} \quad M_y = \int_R x \rho(x, y) \, dA.
\]

If \( m \) is the mass of the lamina, then the center of mass is

\[
(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).
\]

If \( R \) represents a simple plane region rather than a lamina, the point \((\bar{x}, \bar{y})\) is called the centroid of the region.
**Moments of Inertia**

The moments of \( M_x \) and \( M_y \) used in determining the center of mass of a lamina are sometimes called the **first moment** about the \( x \)- and \( y \)-axes. In each case, the moment is the product of a mass times a distance.

\[
M_x = \int \int (y) \rho(x, y) \, dA \quad \text{and} \quad M_y = \int \int (x) \rho(x, y) \, dA
\]

You will now look at another type of moment—the **second moment**, or the **moment of inertia** of a lamina about a line. In the same way that mass is a measure of the tendency of matter to resist a change in straight-line motion, the moment of inertia about a line is a measure of the tendency of matter to resist a change in rotational motion. For example, if a particle of mass \( m \) is a distance \( d \) from a fixed line, its moment of inertia about the line is defined as

\[
I = md^2 = \text{(mass)(distance)}^2.
\]

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the \( x \)- and \( y \)-axes of a lamina of variable density. These second moments are denoted by \( I_x \) and \( I_y \), and in each case the moment is the product of a mass times the square of a distance.

\[
I_x = \int \int (y^2) \rho(x, y) \, dA \quad \text{and} \quad I_y = \int \int (x^2) \rho(x, y) \, dA
\]

The sum of the moments \( I_x \) and \( I_y \) is called the **polar moment of inertia** and is denoted by \( I_p \).

The moment of inertia \( I \) of a revolving lamina can be used to measure its kinetic energy. For example, suppose a planar lamina is revolving about a line with an **angular speed** of \( \omega \) radians per second, as shown in Figure 14.41. The kinetic energy \( E \) of the revolving lamina is

\[
E = \frac{1}{2} I \omega^2. \quad \text{Kinetic energy for rotational motion}
\]

On the other hand, the kinetic energy \( E \) of a mass \( m \) moving in a straight line at a velocity \( v \) is

\[
E = \frac{1}{2} mv^2. \quad \text{Kinetic energy for linear motion}
\]

So, the kinetic energy of a mass moving in a straight line is proportional to its mass, but the kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

The **radius of gyration** \( r \) of a revolving mass \( m \) with moment of inertia \( I \) is defined as

\[
r = \sqrt{\frac{I}{m}}. \quad \text{Radius of gyration}
\]

If the entire mass were located at a distance \( r \) from its axis of revolution, it would have the same moment of inertia and, consequently, the same kinetic energy. For instance, the radius of gyration of the lamina in Example 4 about the \( x \)-axis is given by

\[
r = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32,768k/315}{256k/15}} = \sqrt{\frac{128}{21}} = 2.469.
\]
**DEFINITION OF SURFACE AREA**

If $f$ and its first partial derivatives are continuous on the closed region $R$ in the $xy$-plane, then the area of the surface $S$ given by $z = f(x, y)$ over $R$ is defined as

\[
\text{Surface area} = \iint_R \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2} \, dA.
\]

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

- **Length on $x$-axis:** $\int_a^b ds$
- **Arc length in $xy$-plane:** $\int_a^b \sqrt{1 + \left[f(x)\right]^2} \, dx$
- **Area in $xy$-plane:** $\int_a^b \int_{g_1(x)}^{g_2(x)} \, dy \, dx$
- **Surface area in space:** $\iint_R \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2} \, dA$

**DEFINITION OF TRIPLE INTEGRAL**

If $f$ is continuous over a bounded solid region $Q$, then the **triple integral** of $f$ over $Q$ is defined as

\[
\iiint_Q f(x, y, z) \, dV = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i
\]

provided the limit exists. The **volume** of the solid region $Q$ is given by

\[
\text{Volume of } Q = \iiint_Q \, dV.
\]

**THEOREM 14.4 EVALUATION BY ITERATED INTEGRALS**

Let $f$ be continuous on a solid region $Q$ defined by

\[
a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x, y) \leq z \leq g_2(x, y)
\]

where $h_1$, $h_2$, $g_1$, and $g_2$ are continuous functions. Then,

\[
\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx.
\]
Center of Mass and Moments of Inertia

In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region \( Q \) whose density is given by the density function \( \rho \). The center of mass of a solid region \( Q \) of mass \( m \) is given by \((\bar{x}, \bar{y}, \bar{z})\), where

\[
m = \iiint_{Q} \rho(x, y, z) \, dV \quad \text{Mass of the solid}
\]

\[
M_{xz} = \iiint_{Q} xp(x, y, z) \, dV \quad \text{First moment about \(yz\)-plane}
\]

\[
M_{yz} = \iiint_{Q} yp(x, y, z) \, dV \quad \text{First moment about \(xz\)-plane}
\]

\[
M_{zx} = \iiint_{Q} zp(x, y, z) \, dV \quad \text{First moment about \(xy\)-plane}
\]

and

\[
\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{zx}}{m} \quad \bar{z} = \frac{M_{xy}}{m}
\]

The quantities \( M_{xz}, M_{yz}, \) and \( M_{zx} \) are called the first moments of the region \( Q \) about the \(yz\)-, \(xz\)-, and \(xy\)-planes, respectively.

The first moments for solid regions are taken about a plane, whereas the second moments for solids are taken about a line. The second moments (or moments of inertia) about the \(x\)-, \(y\)-, and \(z\)-axes are as follows.

\[
I_x = \iiint_{Q} (y^2 + z^2)p(x, y, z) \, dV \quad \text{Moment of inertia about \(x\)-axis}
\]

\[
I_y = \iiint_{Q} (x^2 + z^2)p(x, y, z) \, dV \quad \text{Moment of inertia about \(y\)-axis}
\]

\[
I_z = \iiint_{Q} (x^2 + y^2)p(x, y, z) \, dV \quad \text{Moment of inertia about \(z\)-axis}
\]

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

\[
I_x = I_{xz} + I_{xy} \quad I_y = I_{yz} + I_{yx} \quad \text{and} \quad I_z = I_{zx} + I_{yz}
\]

where \(I_{yx}, I_{zx}, \) and \(I_{yz}\) are as follows.

\[
I_{xy} = \iiint_{Q} z^2p(x, y, z) \, dV
\]

\[
I_{xz} = \iiint_{Q} y^2p(x, y, z) \, dV
\]

\[
I_{yz} = \iiint_{Q} x^2p(x, y, z) \, dV
\]