Chapter 14.1 to 15.2 Formulas

AREA OF A REGION IN THE PLANE

1. If $R$ is defined by $a \leq x \leq b$ and $g_1(y) \leq y \leq g_2(y)$, where $g_1$ and $g_2$ are continuous on $[a, b]$, then the area of $R$ is given by

\[ A = \int_a^b \int_{g_1(y)}^{g_2(y)} dy \, dx. \]

Figure 14.2 (vertically simple)

2. If $R$ is defined by $c \leq y \leq d$ and $h_1(x) \leq x \leq h_2(x)$, where $h_1$ and $h_2$ are continuous on $[c, d]$, then the area of $R$ is given by

\[ A = \int_c^d \int_{h_1(x)}^{h_2(x)} dx \, dy. \]

Figure 14.3 (horizontally simple)

DEFINITION OF DOUBLE INTEGRAL

If $f$ is defined on a closed, bounded region $R$ in the $xy$-plane, then the double integral of $f$ over $R$ is given by

\[ \iint_R f(x, y) \, dA = \lim_{\|R\| \to 0} \sum f(x_i, y_i) \, \Delta A. \]

provided the limit exists. If the limit exists, then $f$ is integrable over $R$.

VOLUME OF A SOLID REGION

If $f$ is integrable over a plane region $R$ and $f(x, y) \geq 0$ for all $(x, y)$ in $R$, then the volume of the solid region that lies above $R$ and below the graph of $f$ is defined as

\[ V = \iint_R f(x, y) \, dA. \]

THEOREM 14.2 FUBIN'S THEOREM

Let $f$ be continuous on a plane region $R$.

1. If $R$ is defined by $a \leq x \leq b$ and $g_1(y) \leq y \leq g_2(y)$, where $g_1$ and $g_2$ are continuous on $[a, b]$, then

\[ \iint_R f(x, y) \, dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dy \, dx. \]

2. If $R$ is defined by $c \leq y \leq d$ and $h_1(x) \leq x \leq h_2(x)$, where $h_1$ and $h_2$ are continuous on $[c, d]$, then

\[ \iint_R f(x, y) \, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y) \, dx \, dy. \]
**DEFINITION OF THE AVERAGE VALUE OF A FUNCTION OVER A REGION**

If \( f \) is integrable over the plane region \( R \), then the average value of \( f \) over \( R \) is

\[
\frac{1}{A} \int_R f(x, y) \, dA
\]

where \( A \) is the area of \( R \).

**THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM**

Let \( R \) be a plane region consisting of all points \((x, y) = (r \cos \theta, r \sin \theta)\)

satisfying the conditions \( 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \), where

\[ 0 \leq (\beta - \alpha) \leq 2\pi. \]

If \( g_1 \) and \( g_2 \) are continuous on \([\alpha, \beta]\) and \( f \) is continuous on \(\overline{R} \), then

\[
\int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

**NOTE**

If \( z = f(x, y) \) is nonnegative on \( R \), then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of \( f \) and the region \( R \). When using the integral in Theorem 14.3, be sure not to omit the extra factor of \( r \) in the integrand.

The region \( R \) is restricted to two basic types, \( r \)-simple regions and \( \theta \)-simple regions, as shown in Figure 14.29.

**DEFINITION OF MASS OF A PLANAR LAMINA OF VARIABLE DENSITY**

If \( \rho \) is a continuous density function on the lamina corresponding to a plane region \( R \), then the mass \( m \) of the lamina is given by

\[
m = \int_R \rho(x, y) \, dA.
\]

**MOMENTS AND CENTER OF MASS OF A VARIABLE DENSITY PLANAR LAMINA**

Let \( \rho \) be a continuous density function on the planar lamina \( R \). The moments of mass with respect to the \( x \)- and \( y \)-axes are

\[
M_x = \int_R y\rho(x, y) \, dA \quad \text{and} \quad M_y = \int_R x\rho(x, y) \, dA.
\]

If \( m \) is the mass of the lamina, then the center of mass is

\[
(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).
\]

If \( R \) represents a simple plane region rather than a lamina, the point \((\bar{x}, \bar{y})\) is called the centroid of the region.
Moments of Inertia

The moments of inertia used in determining the center of mass of a lamina are sometimes called the first moments about the x- and y-axes. In each case, the moment is the product of a mass times a distance.

\[ M_x = \int \int (y) \rho(x, y) \, dA \]

Distance to x-axis  \hspace{1cm} \text{Mass}  \hspace{1cm} \text{Distance to y-axis}

\[ M_y = \int \int (x) \rho(x, y) \, dA \]

You will now look at another type of moment—the second moment, or the moment of inertia of a lamina about a line. In the same way that mass is a measure of the tendency of matter to resist a change in straight-line motion, the moment of inertia about a line is a measure of the tendency of matter to resist a change in rotational motion. For example, if a particle of mass \( m \) is a distance \( d \) from a fixed line, its moment of inertia about the line is defined as

\[ I = md^2 = (\text{mass})(\text{distance})^2. \]

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the x- and y-axes of a lamina of variable density. These second moments are denoted by \( I_x \) and \( I_y \), and in each case the moment is the product of a mass times the square of a distance.

\[ I_x = \int \int (y^2) \rho(x, y) \, dA \]

Square of distance to x-axis  \hspace{1cm} \text{Mass}  \hspace{1cm} \text{Square of distance to y-axis}

\[ I_y = \int \int (x^2) \rho(x, y) \, dA \]

The sum of the moments \( I_x \) and \( I_y \) is called the polar moment of inertia and is denoted by \( I_p \).

The moment of inertia \( I \) of a revolving lamina can be used to measure its kinetic energy. For example, suppose a planar lamina is revolving about a line with an angular speed of \( \omega \) radians per second, as shown in Figure 14.41. The kinetic energy \( E \) of the revolving lamina is

\[ E = \frac{1}{2} I \omega^2. \quad \text{Kinetic energy for rotational motion} \]

On the other hand, the kinetic energy \( E \) of a mass \( m \) moving in a straight line at a velocity \( v \) is

\[ E = \frac{1}{2} mv^2. \quad \text{Kinetic energy for linear motion} \]

So, the kinetic energy of a mass moving in a straight line is proportional to its mass, but the kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

The radius of gyration \( r \) of a revolving mass \( m \) with moment of inertia \( I \) is defined as

\[ r = \sqrt{\frac{I}{m}}. \quad \text{Radius of gyration} \]

If the entire mass were located at a distance \( r \) from its axis of revolution, it would have the same moment of inertia and, consequently, the same kinetic energy. For instance, the radius of gyration of the lamina in Example 4 about the \( x \)-axis is given by

\[ r = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32,768k/315}{256k/15}} = \sqrt{\frac{128}{21}} = 2.469. \]

3
DEFINITION OF SURFACE AREA

If \( f \) and its first partial derivatives are continuous on the closed region \( R \) in the \( xy \)-plane, then the area of the surface \( S \) given by \( z = f(x, y) \) over \( R \) is defined as

\[
\text{Surface area} = \iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA.
\]

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

\[
\text{Length on } x\text{-axis: } \int_a^b dx
\]

\[
\text{Arc length in } xy\text{-plane: } \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

\[
\text{Area in } xy\text{-plane: } \iint_D dA
\]

\[
\text{Surface area in space: } \iiint_Q dS = \iiint_Q \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA
\]

DEFINITION OF TRIPLE INTEGRAL

If \( f \) is continuous over a bounded solid region \( Q \), then the triple integral of \( f \) over \( Q \) is defined as

\[
\iiint_Q f(x, y, z) \, dV = \lim_{m \to \infty} \sum_{i=1}^m f(x_i, y_i, z_i) \Delta V_i
\]

provided the limit exists. The volume of the solid region \( Q \) is given by

\[
\text{Volume of } Q = \iiint_Q dV.
\]

THEOREM 14.4 EVALUATION BY ITERATED INTEGRALS

Let \( f \) be continuous on a solid region \( Q \) defined by

\[
a \leq x \leq b, \quad h_1(y) \leq y \leq h_2(y), \quad g_1(x, y) \leq z \leq g_2(x, y)
\]

where \( h_1, h_2, g_1, \) and \( g_2 \) are continuous functions. Then,

\[
\iiint_Q f(x, y, z) \, dV = \int_a^b \left[ \int_{h_1(x)}^{h_2(x)} f(x, y, z) \, dy \right] dx.
\]
Center of Mass and Moments of Inertia

In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region $Q$ whose density is given by the density function $\rho$. The center of mass of a solid region $Q$ of mass $m$ is given by $(\bar{x}, \bar{y}, \bar{z})$, where

$$m = \iiint_{Q} \rho(x, y, z) \, dV$$
Mass of the solid

$$M_{yz} = \iiint_{Q} x \rho(x, y, z) \, dV$$
First moment about $yz$-plane

$$M_{xz} = \iiint_{Q} y \rho(x, y, z) \, dV$$
First moment about $xz$-plane

$$M_{xy} = \iiint_{Q} z \rho(x, y, z) \, dV$$
First moment about $xy$-plane

and

$$\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

The quantities $M_{yz}$, $M_{xz}$, and $M_{xy}$ are called the first moments of the region $Q$ about the $yz$-, $xz$-, and $xy$-planes, respectively.

The first moments for solid regions are taken about a plane, whereas the second moments for solids are taken about a line. The second moments (or moments of inertia) about the $x$-, $y$-, and $z$-axes are as follows.

$$I_x = \iiint_{Q} (y^2 + z^2) \rho(x, y, z) \, dV$$
Moment of inertia about $x$-axis

$$I_y = \iiint_{Q} (x^2 + z^2) \rho(x, y, z) \, dV$$
Moment of inertia about $y$-axis

$$I_z = \iiint_{Q} (x^2 + y^2) \rho(x, y, z) \, dV$$
Moment of inertia about $z$-axis

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

$$I_x = I_{xz} + I_{xy} \quad I_y = I_{yz} + I_{xy} \quad and \quad I_z = I_{xz} + I_{yz}$$

where $I_{xy}$, $I_{xz}$, and $I_{yz}$ are as follows.

$$I_{xy} = \iiint_{Q} z^2 \rho(x, y, z) \, dV$$

$$I_{xz} = \iiint_{Q} y^2 \rho(x, y, z) \, dV$$

$$I_{yz} = \iiint_{Q} x^2 \rho(x, y, z) \, dV$$
**Triple Integrals in Cylindrical Coordinates**

$$\int_Q \int f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{r_1}^{r_2} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta.$$  

**Triple Integrals in Spherical Coordinates**

$$\int_Q \int f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$  

**Jacobian**

**DEFINITION OF THE JACOBIAN**

If \( x = g(u, v) \) and \( y = h(u, v) \), then the Jacobian of \( x \) and \( y \) with respect to \( u \) and \( v \), denoted by \( \frac{\partial(x, y)}{\partial(u, v)} \), is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$  

**Change of Variables for Double Integrals**

**THEOREM 14.5 CHANGE OF VARIABLES FOR DOUBLE INTEGRALS**

Let \( R \) be a vertically or horizontally simple region in the \( xy \)-plane, and let \( S \) be a vertically or horizontally simple region in the \( uv \)-plane. Let \( T \) from \( S \) to \( R \) be given by \( T(u, v) = (x, y) = (g(u, v), h(u, v)) \), where \( g \) and \( h \) have continuous first partial derivatives. Assume that \( T \) is one-to-one except possibly on the boundary of \( S \). If \( f \) is continuous on \( R \), and \( \frac{\partial(x, y)}{\partial(u, v)} \) is nonzero on \( S \), then

$$\int_R f(x, y) \, dx \, dy = \int_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$
**Vector Fields**

**DEFINITION OF VECTOR FIELD**

A **vector field** over a **plane region** $R$ is a function $F(x, y)$ that assigns a vector $F(x, y)$ to each point in $R$.

A **vector field** over a **solid region** $Q$ in space is a function $F$ that assigns a vector $F(x, y, z)$ to each point in $Q$.

**DEFINITION OF INVERSE SQUARE FIELD**

Let $r(t) = xi + yj + zk$ be a position vector. The vector field $F$ is an inverse square field if

$$F(x, y, z) = \frac{k}{||r||^3} \hat{u}$$

where $k$ is a real number and $\hat{u} = r/||r||$ is a unit vector in the direction of $r$.

**DEFINITION OF CONSERVATIVE VECTOR FIELD**

A vector field $F$ is called **conservative** if there exists a differentiable function $f$ such that $F = \nabla f$. The function $f$ is called the **potential function** for $F$.

**THEOREM 15.1 TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE**

Let $M$ and $N$ have continuous first partial derivatives on an open disk $R$. The vector field given by $F(x, y) = Mi + Nj$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

**DEFINITION OF CURL OF A VECTOR FIELD**

The curl of $F(x, y, z) = Mi + Nj +Pk$ is

$$\text{curl } F(x, y, z) = \nabla \times F(x, y, z) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M & N & P
\end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} - \left(\frac{\partial P}{\partial z} - \frac{\partial M}{\partial x}\right)\hat{j} + \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)\hat{k}.$$

**THEOREM 15.2 TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE**

Suppose that $M$, $N$, and $P$ have continuous first partial derivatives in an open sphere $Q$ in space. The vector field given by $F(x, y, z) = Mi + Nj +Pk$ is conservative if and only if

$$\text{curl } F(x, y, z) = 0.$$

That is, $F$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial M}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
DEFINITION OF DIVERGENCE OF A VECTOR FIELD

The divergence of \( \mathbf{F}(x, y) = Mi + NJ \) is

\[
\text{div} \, \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.
\]

Plane

The divergence of \( \mathbf{F}(x, y, z) = Mi + NJ + PK \) is

\[
\text{div} \, \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

Space

If \( \text{div} \, \mathbf{F} = 0 \), then \( \mathbf{F} \) is said to be divergence free.

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \leq t \leq b \]

is smooth if

\[
\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt} \quad \frac{dz}{dt}.
\]

are continuous on \([a, b]\) and not simultaneously 0 on \((a, b)\). Similarly, a space curve \( \mathbf{C} \) given by

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b \]

is smooth if

\[
\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}
\]

are continuous on \([a, b]\) and not simultaneously 0 on \((a, b)\). A curve \( C \) is piecewise smooth if the interval \([a, b]\) can be partitioned into a finite number of subintervals, on each of which \( C \) is smooth.
DEFINITION OF LINE INTEGRAL

If \( f \) is defined in a region containing a smooth curve \( C \) of finite length, then the **line integral of \( f \) along \( C \)** is given by

\[
\int_C f(x, y) \, ds = \lim_{||\Delta||\to 0} \sum_{i=1}^n f(x_i, y_i) \Delta s_i \quad \text{Plane}
\]

or

\[
\int_C f(x, y, z) \, ds = \lim_{||\Delta||\to 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta s_i \quad \text{Space}
\]

provided this limit exists.

To evaluate a line integral over a plane curve \( C \) given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \), use the fact that

\[ ds = ||\mathbf{r}'(t)|| \, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt. \]

A similar formula holds for a space curve, as indicated in Theorem 15.4.

**THEOREM 15.4 EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL**

Let \( f \) be continuous in a region containing a smooth curve \( C \). If \( C \) is given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \), where \( a \leq t \leq b \), then

\[
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.
\]

If \( C \) is given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \), where \( a \leq t \leq b \), then

\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]

Note that if \( f(x, y, z) = 1 \), the line integral gives the area length of the curve \( C \), as defined in Section 12.5. That is,

\[
\int_C 1 \, ds = \int_a^b ||\mathbf{r}'(t)|| \, dt = \text{length of curve } C.
\]
To Compute Work

**Definition of the Line Integral of a Vector Field**

Let \( \mathbf{F} \) be a continuous vector field defined on a smooth curve \( C \) given by \( \mathbf{r}(t) \), \( a \leq t \leq b \). The line integral of \( \mathbf{F} \) on \( C \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt.
\]

### Line Integrals in Differential Form

This differential form can be extended to three variables. The parentheses are often omitted, as follows.

\[
\int_C M \, dx + N \, dy \quad \text{and} \quad \int_C M \, dx + N \, dy + P \, dz
\]

**Note**. The orientation of \( C \) affects the value of the differential form of a line integral. Specifically, if \( -C \) has the orientation opposite to that of \( C \), then

\[
\int_C M \, dx + N \, dy = -\int_C M \, dx + N \, dy.
\]

So, of the three line integral forms presented in this section, the orientation of \( C \) does not affect the form \( \int_C f(x, y) \, ds \), but it does affect the vector form and the differential form.