Section 11.3

Work

The work \( W \) done by the constant force \( \mathbf{F} \) acting along the line of motion of an object is given by

\[
W = (\text{magnitude of force})(\text{distance}) = |\mathbf{F}| |\mathbf{PQ}| \cos \theta
\]

as shown in Figure 11.33(a). If the constant force \( \mathbf{F} \) is not directed along the line of motion, you can see from Figure 11.33(b) that the work \( W \) done by the force is

\[
W = |\text{proj}_P \mathbf{F}| |\mathbf{PQ}| = (\cos \theta |\mathbf{F}|) |\mathbf{PQ}| = \mathbf{F} \cdot \mathbf{PQ}.
\]

This notion of work is summarized in the following definition.

**Definition of Work**

The work \( W \) done by a constant force \( \mathbf{F} \) at its point of application moves along the vector \( \mathbf{PQ} \) is given by either of the following:

1. \( W = |\text{proj}_P \mathbf{F}| |\mathbf{PQ}| \)
2. \( W = \mathbf{F} \cdot \mathbf{PQ} \)

\[
\mathbf{u} \cdot \mathbf{v} = (\cos \theta) |\mathbf{u}| |\mathbf{v}| \cos \theta = \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \cos \theta \nu_1 \cos \theta \end{pmatrix} = \cos \theta |\mathbf{u}| |\mathbf{v}| \cos \theta = \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \cos \theta + v_1 \cos \theta \end{pmatrix} = 375 \sqrt{3} \text{ ft-lb}
\]

Try It 7

A toy wagon is pulled by exerting a force of 15 pounds on a handle that makes a 30° angle with the horizontal (see figure). Find the work done in pulling the wagon 50 feet.

\[
\text{Work} = \mathbf{F} \cdot \mathbf{PQ}
\]

\[
\mathbf{F} = 15 (\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})
\]

\[
\mathbf{PQ} = 50 \mathbf{i} + 0 \mathbf{j}
\]

\[
\mathbf{F} \cdot \mathbf{PQ} = \left( \frac{15 \sqrt{3}}{2} \mathbf{i}, \frac{15}{2} \mathbf{j} \right) \cdot (50 \mathbf{i}, 0 \mathbf{j})
\]

\[
= \left( \frac{15 \sqrt{3}}{2} \right) (50) + (\frac{15}{2})(0)
\]

\[
= 375 \sqrt{3} \text{ ft-lb}
\]
11.4 Exercises

In Exercises 1–6, find the cross product of the unit vectors and sketch your result.

1. \( \mathbf{j} \times \mathbf{i} \)  
2. \( \mathbf{i} \times \mathbf{j} \)  
3. \( \mathbf{j} \times \mathbf{k} \)  
4. \( \mathbf{k} \times \mathbf{j} \)  
5. \( \mathbf{i} \times \mathbf{k} \)  
6. \( \mathbf{k} \times \mathbf{i} \)

In Exercises 7–10, find (a) \( \mathbf{u} \times \mathbf{v} \), (b) \( \mathbf{v} \times \mathbf{u} \), and (c) \( \mathbf{v} \times \mathbf{v} \).

7. \( \mathbf{u} = 2\mathbf{i} + 4\mathbf{j} \)  
8. \( \mathbf{u} = 3\mathbf{i} + 5\mathbf{k} \)  
9. \( \mathbf{u} = (7, 3, 2) \)  
10. \( \mathbf{u} = (3, 2, -1) \)

In Exercises 11–16, find \( \mathbf{u} \times \mathbf{v} \) and show that it is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

11. \( \mathbf{u} = (12, -3, 0) \)  
12. \( \mathbf{u} = (-1, 1, 2) \)  
13. \( \mathbf{u} = (2, -3, 1) \)  
14. \( \mathbf{u} = (-10, 0, 6) \)  
15. \( \mathbf{u} = 1 + \mathbf{j} + \mathbf{k} \)  
16. \( \mathbf{u} = 1 + 6\mathbf{j} \)

In Exercises 17–20, use the vectors \( \mathbf{u} \) and \( \mathbf{v} \) shown in the figure to sketch a vector in the direction of the indicated cross product in a right-handed system.

17. \( \mathbf{u} \times \mathbf{v} \)  
18. \( \mathbf{v} \times \mathbf{u} \)  
19. \( (-\mathbf{v}) \times \mathbf{u} \)  
20. \( \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) \)

In Exercises 21–24, use a computer algebra system to find \( \mathbf{u} \times \mathbf{v} \) and a unit vector orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \).

21. \( \mathbf{u} = (4, -3, 5, 7) \)  
22. \( \mathbf{u} = (-8, -6, 4, 5) \)  
23. \( \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \)  
24. \( \mathbf{u} = 0.7\mathbf{k} \)

25. Programming Given the vectors \( \mathbf{u} \) and \( \mathbf{v} \) in component form, write a program for a graphing utility in which the output is \( \mathbf{u} \times \mathbf{v} \) and \( |\mathbf{u} \times \mathbf{v}| \).

26. Programming Use the program you wrote in Exercise 25 to find \( \mathbf{u} \times \mathbf{v} \) and \( |\mathbf{u} \times \mathbf{v}| \) for \( \mathbf{u} = (-2, 6, 10) \) and \( \mathbf{v} = (3, 8, 5) \).

Area In Exercises 27–30, find the area of the parallelogram that has the given vectors as adjacent sides. Use a computer algebra system or a graphing utility to verify your result.

27. \( \mathbf{u} = \mathbf{j} \)  
28. \( \mathbf{u} = \mathbf{i} + \mathbf{j} \)  
29. \( \mathbf{u} = (3, 2, -1) \)  
30. \( \mathbf{u} = (2, -1, 0) \)

Area In Exercises 31 and 32, verify that the points are the vertices of a parallelogram, and find its area.

31. \( A(0, 3, 2), B(1, 5, 5), C(-6, 9, 5), D(5, 7, 2) \)  
32. \( A(2, -3, 1), B(6, 5, -1), C(7, 2, 2), D(3, -6, 4) \)

Area In Exercises 33–36, find the area of the triangle with the given vertices. (Hint: \( \frac{1}{2} |\mathbf{u} \times \mathbf{v}| \) is the area of the triangle having \( \mathbf{u} \) and \( \mathbf{v} \) as adjacent sides.)

33. \( A(0, 0, 0), B(1, 0, 3), C(-3, -2, 0) \)  
34. \( A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0) \)  
35. \( A(-2, -7, 3), B(-1, 5, 8), C(4, 6, -1) \)  
36. \( A(1, 2, 0), B(-2, 1, 0), C(0, 0, 0) \)

37. Torque A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a 40° angle with the horizontal (see figure). The crank is 6 inches in length. Find the torque at \( P \).

38. Torque Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

39. Optimization A force of 56 pounds acts on the pipe wrench shown in the figure on the next page.

(a) Find the magnitude of the moment about \( O \) by evaluating \( |\mathbf{OA} \times \mathbf{F}| \). Use a graphing utility to graph the resulting function of \( \theta \).

(b) Use the result of part (a) to determine the magnitude of the moment when \( \theta = 45^\circ \).

(c) Use the result of part (a) to determine the angle \( \theta \) when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?
40. Optimization A force of 180 pounds acts on the bracket shown in the figure.
(a) Determine the vector $\overrightarrow{AB}$ and the vector $\mathbf{F}$ representing the force. (F will be in terms of $\theta$.)
(b) Find the magnitude of the moment about $A$ by evaluating $|\overrightarrow{AB} \times \mathbf{F}|$.
(c) Use the result of part (b) to determine the magnitude of the moment when $\theta = 30^\circ$.
(d) Use the result of part (b) to determine the angle $\theta$ when the magnitude of the moment is maximum. At that angle, what is the relationship between the vectors $\mathbf{F}$ and $\overrightarrow{AB}$? Is it what you expected? Why or why not?
(e) Use a graphing utility to graph the function for the magnitude of the moment about $A$ for $0^\circ \leq \theta \leq 180^\circ$. Find the zero of the function in the given domain. Interpret the meaning of the zero in the context of the problem.

In Exercises 41–44, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
41. $\mathbf{u} = (1, 1, 1)$
\hspace{1cm} $\mathbf{v} = (2, 1, 0)$
\hspace{1cm} $\mathbf{w} = (0, 3, 2)$

42. $\mathbf{u} = (1, 1, 1)$
\hspace{1cm} $\mathbf{v} = (2, 1, 0)$
\hspace{1cm} $\mathbf{w} = (0, 3, 2)$

43. $\mathbf{v} = \mathbf{k}$
\hspace{1cm} $\mathbf{w} = (0, 3, 2)$

44. $\mathbf{v} = \mathbf{k}$
\hspace{1cm} $\mathbf{w} = (0, 3, 2)$

In Exercises 45 and 46, use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
45. $\mathbf{u} = \mathbf{i} + \mathbf{j}$
\hspace{1cm} $\mathbf{v} = \mathbf{j} + \mathbf{k}$
\hspace{1cm} $\mathbf{w} = \mathbf{i} + \mathbf{k}$

46. $\mathbf{u} = (1, 3, 1)$
\hspace{1cm} $\mathbf{v} = (0, 6, 0)
\hspace{1cm} $\mathbf{w} = (-3, 0, -3)$

Volume In Exercises 47 and 48, find the volume of the parallelepiped with the given vertices.
47. $(0, 0, 0), (3, 0, 0), (0, 5, 1), (2, 0, 5)$
\hspace{1cm} $(3, 5, 1), (5, 0, 5), (2, 5, 6), (5, 5, 6)$

48. $(0, 0, 0), (0, 4, 0), (3, 0, 0), (-1, 1, 5)$
\hspace{1cm} $(-3, 4, 0), (-1, 5, 5), (-4, 1, 5), (-4, 5, 5)$

49. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, what can you conclude about $\mathbf{u}$ and $\mathbf{v}$?

50. Identify the dot products that are equal. Explain your reasoning. (Assume $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors.)
(a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
\hspace{1cm} (b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
\hspace{1cm} (c) $\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}$
\hspace{1cm} (d) $(\mathbf{u} \cdot \mathbf{w}) \cdot \mathbf{v}$
\hspace{1cm} (e) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
\hspace{1cm} (f) $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$
\hspace{1cm} (g) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
\hspace{1cm} (h) $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$

51. Define the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$.

52. State the geometric properties of the cross product.

53. If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change? Explain.

54. The vertices of a triangle in space are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$. Explain how to find a vector perpendicular to the triangle.

55. True or False? In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.
56. It is possible to find the cross product of two vectors in a two-dimensional coordinate system.

57. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$.

58. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$.

59. In Exercises 59–66, prove the property of the cross product.
59. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
60. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
61. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
62. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
63. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$
64. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
65. Prove that $[n \times \mathbf{v}] = [n \times \mathbf{w}]$ if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
66. Prove that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}$.
Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section you will study a product that will yield such a vector. It is called the cross product, and it is most conveniently defined and calculated using the standard unit vector form. Because the cross product yields a vector, it is also called the vector product.

<table>
<thead>
<tr>
<th>DEFINITION OF CROSS PRODUCT OF TWO VECTORS IN SPACE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} ) and ( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} ) be vectors in space. The cross product of ( \mathbf{u} ) and ( \mathbf{v} ) is the vector ( \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} ).</td>
</tr>
</tbody>
</table>
A convenient way to calculate \( \mathbf{u} \times \mathbf{v} \) is to use the following **determinant form** with cofactor expansion. (This \( 3 \times 3 \) determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
i & j & k \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 
\end{vmatrix}
\]

Put "i" in Row 2.

Put "j" in Row 3.

\[
\begin{align*}
&= i(1) - j(-1) + k(-1) \\
&= (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k
\end{align*}
\]

Note the minus sign in front of the \( \mathbf{j} \)-component. Each of the three \( 2 \times 2 \) determinants can be evaluated by using the following diagonal pattern.

\[
\begin{vmatrix}
a & b \\
c & d 
\end{vmatrix} = ad - bc
\]

Here are a couple of examples.

\[
\begin{align*}
\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} &= (2)(-1) - (4)(3) = -2 - 12 = -14
\end{align*}
\]
A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following determinant form with cofactor expansion. (This $3 \times 3$ determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
  i & j & k \\
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
\end{vmatrix}
$$

- Put "u" in Row 2.
- Put "v" in Row 3.

$$
= \begin{vmatrix}
  i & j & k \\
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
\end{vmatrix} + \begin{vmatrix}
  i & j & k \\
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
\end{vmatrix}
$$

$$
= (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k
$$

Note the minus sign in front of the $j$-component. Each of the three $2 \times 2$ determinants can be evaluated by using the following diagonal pattern.

$$
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix} = ad - bc
$$

Here are a couple of examples.

$$
\begin{vmatrix}
  2 & 4 \\
  3 & -1 \\
\end{vmatrix} = (2)(-1) - (4)(3) = -2 - 12 = -14
$$
Try It 1

Given \( \mathbf{u} = 6\mathbf{i} - \mathbf{j} + 4\mathbf{k} \) and \( \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 8\mathbf{k} \), find each of the following.

a. \( \mathbf{u} \times \mathbf{v} \)  
\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & -1 & 4 \\
2 & 3 & -8
\end{vmatrix} = \left(\begin{array}{c}
\begin{vmatrix}
-1 & 4 \\
2 & -8
\end{vmatrix} \\
\begin{vmatrix}
6 & 4 \\
2 & 3
\end{vmatrix}
\end{array}\right) \mathbf{i} + \left(\begin{array}{c}
\begin{vmatrix}
6 & 4 \\
2 & 3
\end{vmatrix}
\end{array}\right) \mathbf{j} + \left(\begin{array}{c}
\begin{vmatrix}
-1 & 4 \\
2 & -8
\end{vmatrix}
\end{array}\right) \mathbf{k}
\]
\[
= (8 - 12) \mathbf{i} - (-48 + 8) \mathbf{j} + (18 - 2) \mathbf{k}
\]
\[
= -4\mathbf{i} + 56\mathbf{j} + 20\mathbf{k}
\]

b. \( \mathbf{v} \times \mathbf{u} \)  
\[
\mathbf{v} \times \mathbf{u} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 3 & -8 \\
6 & -1 & 4
\end{vmatrix} = \left(\begin{array}{c}
\begin{vmatrix}
3 & -8 \\
-1 & 4
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
2 & -8 \\
6 & -1
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
2 & 3 \\
6 & -1
\end{vmatrix} \mathbf{k}
\end{array}\right)
\]
\[
= (12 - 8) \mathbf{i} - (8 + 4) \mathbf{j} + (-2 - 8) \mathbf{k}
\]
\[
= 4\mathbf{i} - 56\mathbf{j} - 20\mathbf{k}
\]

c. \( \mathbf{v} \times \mathbf{v} \)  
\[
\mathbf{v} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 3 & -8 \\
2 & 3 & -8
\end{vmatrix} = \left(\begin{array}{c}
\begin{vmatrix}
3 & -8 \\
2 & -8
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
2 & -8 \\
2 & 3
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
2 & 3 \\
2 & 3
\end{vmatrix} \mathbf{k}
\end{array}\right)
\]
\[
= (-24 + 24) \mathbf{i} - (-16 + 16) \mathbf{j} + (6 - 6) \mathbf{k}
\]
\[
= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}
\]
THEOREM 11.7 ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors in space, and let \( c \) be a scalar.

1. \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)
2. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \)
3. \( c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) \)
4. \( \mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0} \)
5. \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \)
6. \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)
Vectors and the Geometry of Space

Note that Property 1 of Theorem 11.7 indicates that the cross product is not commutative. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following theorem lists some other geometric properties of the cross product of two vectors.

**THEOREM 11.8 GEOMETRIC PROPERTIES OF THE CROSS PRODUCT**

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let $\theta$ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $|\mathbf{u} \times \mathbf{v}| = \text{area of parallelogram having } \mathbf{u} \text{ and } \mathbf{v} \text{ as adjacent sides.}$

**PROOF** To prove Property 2, note because $\cos \theta = (\mathbf{u} \cdot \mathbf{v})/(|\mathbf{u}| |\mathbf{v}|)$, it follows that

$$|\mathbf{u}| |\mathbf{v}| \sin \theta = |\mathbf{u}| |\mathbf{v}| \sqrt{1 - \cos^2 \theta} = |\mathbf{u}| |\mathbf{v}| \sqrt{1 - (\mathbf{u} \cdot \mathbf{v})^2/|\mathbf{u}|^2 |\mathbf{v}|^2}.$$
The vectors $\mathbf{u}$ and $\mathbf{v}$ form adjacent sides of a parallelogram. Figure 11.35

To prove Property 2, note because $\cos \theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\| \|\mathbf{v}\|)$, it follows that

$$\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - \cos^2 \theta}$$

$$= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - (\mathbf{u} \cdot \mathbf{v})^2/\|\mathbf{u}\|^2 \|\mathbf{v}\|^2}$$

$$= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}$$

$$= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2}$$

$$= \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2}$$

$$= \|\mathbf{u} \times \mathbf{v}\|.$$
Both \( \mathbf{u} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{u} \) are perpendicular to the plane determined by \( \mathbf{u} \) and \( \mathbf{v} \). One way to remember the orientations of the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{u} \times \mathbf{v} \) is to compare them with the unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} = \mathbf{i} \times \mathbf{j} \), as shown in Figure 11.36. The three vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{u} \times \mathbf{v} \) form a right-handed system, whereas the three vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{v} \times \mathbf{u} \) form a left-handed system.

![Diagram of vectors and cross products](image)

**Right-handed systems**

**Figure 11.36**

\[
\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{i} \times \mathbf{j} = \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}
\]

\[
\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}
\]

\[
\mathbf{i} \times \mathbf{j} = \begin{vmatrix} 0 & 0 & \mathbf{i} \\ 0 & \mathbf{j} & 0 \\ \mathbf{k} & 0 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 1\mathbf{k}
\]
Try It 2

Find a unit vector that is orthogonal to both

\( u = j + 6k \) and \( v = i - 2j + k \).

\[
\begin{align*}
\mathbf{u} = \langle 0, 1, 6 \rangle & \quad \mathbf{v} = \langle 1, -2, 1 \rangle \\
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} i & j & k \\ 0 & 1 & 6 \\ 1 & -2 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 6 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 0 & 6 \\ 1 & -2 \end{vmatrix} j + \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} k \\
&= 13i + 6j - 1k \\
\left\| \mathbf{u} \times \mathbf{v} \right\| &= \sqrt{(13)^2 + (6)^2 + (-1)^2} = \sqrt{206} \\
\frac{1}{\left\| \mathbf{u} \times \mathbf{v} \right\|} \langle 13, 6, -1 \rangle &= \text{Unit Vector} \\
\frac{1}{\sqrt{206}} \langle 13, 6, -1 \rangle &= \langle \frac{13}{\sqrt{206}}, \frac{6}{\sqrt{206}}, -\frac{1}{\sqrt{206}} \rangle
\end{align*}
\]
Try It 3

Parallelogram:

\[ \vec{AD} = -\vec{CB} \quad \text{Parallel} \]
\[ \vec{AB} = -\vec{CD} \quad \text{Parallel} \]

Magnitude of the Cross Product:

\[ ||\vec{AB} \times \vec{AD}|| = \begin{vmatrix} i & j & k \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} = \]
\[ \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} i - \begin{vmatrix} 3 & 3 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} k \]
\[ \vec{AB} \times \vec{AD} = -6i - 6j + 10k \]

\[ ||\vec{AB} \times \vec{AD}|| = \sqrt{(-6)^2 + (-6)^2 + (10)^2} \]
\[ = \sqrt{122} \approx 11.06 \text{ units} \]
In physics, the cross product can be used to measure torque—the moment $M$ of a force $F$ about a point $P$, as shown in Figure 11.39. If the point of application of the force is $O$, the moment of $F$ about $P$ is given by

$$M = \overrightarrow{PQ} \times F.$$  

The magnitude of the moment $M$ measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector $M$.

**EXAMPLE 4** An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a one-foot lever that is attached
Try It 4

A vertical force of 65 pounds is applied to the end of a one-foot lever that is attached to an axle at a point \( P \), as shown in the figure. Find the moment of this force about the point \( P \) when \( \theta = 45^\circ \).

\[
\vec{F} = 0i + 0j - 65k
\]
\[
\vec{PQ} = \hat{1} (\cos 45^\circ j + \sin 45^\circ k)
\]
\[
\vec{F} = \langle 0, 0, -65 \rangle
\]
\[
\vec{PQ} = \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle
\]

Moment = \( \vec{PQ} \times \vec{F} = \begin{vmatrix} i & j & k \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & -65 \end{vmatrix} \)

\[
\begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -65 \end{vmatrix} i - \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & 65 \end{vmatrix} j + \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{vmatrix} k
\]

\[
= -\frac{65 \sqrt{2}}{2} i = -45.96 i
\]

\[
\| \vec{PQ} \times \vec{F} \| = \sqrt{(-45.96)^2 + 0^2 + 0^2} = 45.96 \text{ ft-lbs}
\]
The Triple Scalar Product

For vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in space, the dot product of \( \mathbf{u} \) and \( \mathbf{v} \times \mathbf{w} \)

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})
\]

is called the **triple scalar product**, as defined in Theorem 11.9. The proof of this theorem is left as an exercise (see Exercise 67).

**Theorem 11.9 The Triple Scalar Product**

For \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}, \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}, \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}, \) the triple scalar product is given by

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\]

**Note.** The value of a determinant is multiplied by \(-1\) if two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. So, the following triple scalar products are equivalent.

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})
\]
If the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ do not lie in the same plane, the triple scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron of whose faces are parallelograms) with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges, as Figure 11.41. This is established in the following theorem.

**Theorem 11.10 Geometric Property of the Triple Scalar Product**

The volume $V$ of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$ 

**Proof.** In Figure 11.41, note that

$$(\mathbf{u} \cdot \mathbf{w}) \parallel = \text{area of base}.$$
**EXAMPLE 5** Volume by the Triple Scalar Product

Find the volume of the parallelepiped shown in Figure 11.42. Let \( \mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{j} - 2\mathbf{k}, \) and \( \mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} \) be adjacent edges.

**Solution** By Theorem 11.10, you have

\[
V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \quad \text{Triple scalar product}
\]

\[
= \begin{vmatrix}
3 & -5 & 1 \\
0 & 2 & -2 \\
3 & 1 & 1
\end{vmatrix} 
= \begin{vmatrix}
2 & -2 \\
1 & 1
\end{vmatrix}(-5) + (1)\begin{vmatrix}
0 & -2 \\
3 & 1
\end{vmatrix} 
= 3(4) + 5(6) + 1(-6)
\]

\[
= 36. \quad \text{Volume}
\]

A natural consequence of Theorem 11.10 is that the volume of the parallelepiped is given by the absolute value of the scalar triple product of the vectors representing its edges.
Try It 5

Find the volume of the parallelepiped having \( u = i + 3j + k \), \( v = 5j + 5k \), and \( w = 4i + 4k \) as adjacent edges (see figure).

**Volume**

\[
V = \frac{1}{6} \left| u \cdot (v \times w) \right|
\]

**Solution**

\[
u \times v = \begin{vmatrix}
i & j & k \\
1 & 3 & 0 \\
0 & 0 & 5
\end{vmatrix}
\]

\[
v \times w = \begin{vmatrix}
i & j & k \\
0 & 0 & 5 \\
4 & 4 & 0
\end{vmatrix}
\]

\[
\left| u \cdot (v \times w) \right| = \sqrt{10}
\]