14.6 Triple Integrals and Applications

- Use a triple integral to find the volume of a solid region.
- Find the center of mass and moments of inertia of a solid region.

Triple Integrals

The procedure used to define a triple integral follows that used for double integrals. Consider a function \( f \) of three variables that is continuous over a bounded solid region \( Q \). Then, encompass \( Q \) with a network of boxes and form the inner partition consisting of all boxes lying entirely within \( Q \), as shown in Figure 14.52. The volume of the \( i \)th box is

\[
\Delta V_i = \Delta x_i \Delta y_i \Delta z_i,
\]

**Volume of \( i \)th box**

The norm \( ||\Delta|| \) of the partition is the length of the longest diagonal of the \( n \) boxes in the partition. Choose a point \((x_p, y_p, z_p)\) in each box and form the Riemann sum

\[
\sum_{i=1}^{n} f(x_p, y_p, z_p) \Delta V_i,
\]

Taking the limit as \( ||\Delta|| \to 0 \) leads to the following definition.

DEFINITION OF TRIPLE INTEGRAL

If \( f \) is continuous over a bounded solid region \( Q \), then the triple integral of \( f \) over \( Q \) is defined as

\[
\iiint_Q f(x, y, z) \, dV = \lim_{||\Delta||\to0} \sum_{i=1}^{n} f(x_p, y_p, z_p) \Delta V_i
\]

provided the limit exists. The **volume** of the solid region \( Q \) is given by

\[
\text{Volume of } Q = \iiint_Q dV.
\]

Some of the properties of double integrals in Theorem 14.1 can be restated in terms of triple integrals.

1. \[
\iiint_Q cf(x, y, z) \, dV = c \iiint_Q f(x, y, z) \, dV
\]

2. \[
\iiint_Q [f(x, y, z) \pm g(x, y, z)] \, dV = \iiint_Q f(x, y, z) \, dV \pm \iiint_Q g(x, y, z) \, dV
\]

3. \[
\iiint_Q f(x, y, z) \, dV = \iiint_{Q_1} f(x, y, z) \, dV + \iiint_{Q_2} f(x, y, z) \, dV
\]

In the properties above, \( Q \) is the union of two nonoverlapping solid subregions \( Q_1 \) and \( Q_2 \). If the solid region \( Q \) is simple, the triple integral \( \iiint_Q f(x, y, z) \, dV \) can be evaluated with an iterated integral using one of the six possible orders of integration:

\[
\begin{align*}
&dx \, dy \, dz, \quad dy \, dx \, dz, \quad dz \, dx \, dy, \\
&dx \, dz \, dy, \quad dy \, dz \, dx, \quad dz \, dy \, dx.
\end{align*}
\]
The following version of Fubini's Theorem describes a region that is considered simple with respect to the order \( dz \, dy \, dx \). Similar descriptions can be given for the other five orders.

**Theorem 14.4 Evaluation by Iterated Integrals**

Let \( f \) be continuous on a solid region \( Q \) defined by

\[
\begin{align*}
 a \leq x \leq b, & \quad h_1(x) \leq y \leq h_2(x), & \quad g_1(x, y) \leq z \leq g_2(x, y)
\end{align*}
\]

where \( h_1, h_2, g_1, \) and \( g_2 \) are continuous functions. Then,

\[
\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx.
\]

To evaluate a triple iterated integral in the order \( dz \, dy \, dx \), hold both \( x \) and \( y \) constant for the innermost integration. Then, hold \( x \) constant for the second integration.

**Try It 1**

Evaluate the triple iterated integral

\[
\int_0^4 \int_0^1 x \sin y \, dz \, dy \, dx,
\]

\[
\begin{align*}
&= \left[ z (x \sin y) \right]_0^{1-x} \\
&= (1-x)(x \sin y) - 0 \\
&= (x \sin y - x^2 \sin y) \, dy
\end{align*}
\]

\[
\int_0^\pi (-x \cos y + x^2 \cos y) \, dy
\]

\[
\left[ -x \cos y + x^2 \cos y \right]_0^\pi
\]

\[
(-x \cos \pi + x^2 \cos \pi) - (-x \cos 0 + x^2 \cos 0)
\]

\[
-x - x^2 + x - x^2
\]

\[
\int_0^4 (2x - 2x^2) \, dx
\]

\[
x^2 - \frac{2}{3} x^3 \bigg|_0^4 = 16 - \frac{2}{3} \left( \frac{64}{1} \right) - 0
\]

\[
\frac{16}{1} - \frac{128}{3} = \frac{48 - 128}{3} = \frac{-80}{3}
\]
Example 1 demonstrates the integration order \( dz \ dy \ dx \). For other orders, you can follow a similar procedure. For instance, to evaluate a triple iterated integral in the order \( dx \ dy \ dz \), hold both \( y \) and \( z \) constant for the innermost integration and integrate with respect to \( x \). Then, for the second integration, hold \( z \) constant and integrate with respect to \( y \). Finally, for the third integration, integrate with respect to \( z \).

To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables. Then, by projecting the solid \( Q \) onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals. For instance, to evaluate

\[
\iiint_{Q} f(x, y, z) \, dz \, dy \, dx
\]

first determine the limits for \( z \), and then the integral has the form

\[
\int \left[ \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \right] \, dy \, dx.
\]

By projecting the solid \( Q \) onto the \( xy \)-plane, you can determine the limits for \( x \) and \( y \) as you did for double integrals, as shown in Figure 14.53.

Solid region \( Q \) lies between two surfaces.
Figure 14.53
Try It 2

Find the volume of the paraboloid solid bounded by the graphs of $z = 9 - x^2 - y^2$ and $z = 0$.

Paraboloid:
$z = 9 - x^2 - y^2$

Symmetry
\[ x^2 + y^2 = 9 \]

Circle of $r = 3$

Projection

\[
\iint_D (9 - x^2 - y^2) \, dy \, dx
\]

Go To Polar

\[
\int_0^3 \int_0^{\sqrt{9-r^2}} (9 - r^2) (r) \, dr \, d\theta
\]

\[
\frac{9r^2}{2} - \frac{r^4}{4} \bigg|_0^3
\]

\[
\frac{81}{4} \int_0^{\pi/2} \, d\theta = \frac{81}{4} \theta \bigg|_0^{\pi/2} = \frac{81\pi}{8} - 0
\]

\[
\frac{81\pi}{8} \quad \text{cubic units}
\]

\[
4 \left( \frac{81\pi}{8} \right) = \frac{81\pi}{2} \quad \text{cubic units}
\]
Example 2 is unusual in that all six possible orders of integration produce integrals of comparable difficulty. Try setting up some other possible orders of integration to find the volume of the ellipsoid. For instance, the order $dx\ dy\ dz$ yields the integral

$$V = 8 \int_{0}^{4} \int_{0}^{\sqrt{16-z^2/2}} \int_{0}^{\sqrt{16-4y^2-z^2/2}} dx\ dy\ dz.$$ 

If you solve this integral, you will obtain the same volume obtained in Example 2. This is always the case—the order of integration does not affect the value of the integral. However, the order of integration often does affect the complexity of the integral. In Example 3, the given order of integration is not convenient, so you can change the order to simplify the problem.
Try It 3

Evaluate

\[
\int_{0}^{\frac{\pi}{6}} \int_{0}^{\sin x} \int_{0}^{\sqrt{\sin x - y^2}} dz \, dy \, dx
\]

by first changing the order of integration to \( dz \, dy \, dx \).
Try It 4

Set up a triple integral for the volume of the solid region bounded by \( z = 4 - x^2 \), \( z = 0 \), \( y = 0 \), and \( y = x \).
Center of Mass and Moments of Inertia

In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region \( Q \) whose density is given by the density function \( \rho \). The center of mass of a solid region \( Q \) of mass \( m \) is given by \((\overline{x}, \overline{y}, \overline{z})\), where

\[
m = \iiint_Q \rho(x, y, z) \, dV
\]

Mass of the solid

\[
M_{xy} = \iiint_Q x \rho(x, y, z) \, dV
\]

First moment about \( yz \)-plane

\[
M_{xz} = \iiint_Q y \rho(x, y, z) \, dV
\]

First moment about \( xz \)-plane

\[
M_{yz} = \iiint_Q z \rho(x, y, z) \, dV
\]

First moment about \( xy \)-plane

and

\[
\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}.
\]

Center of Mass
The quantities $M_{x}$, $M_{y}$, and $M_{z}$ are called the first moments of the region $Q$ about the $yz-$, $xz-$, and $xy-$planes, respectively.

The first moments for solid regions are taken about a plane, whereas the second moments for solids are taken about a line. The second moments (or moments of inertia) about the $x$-, $y$-, and $z$-axes are as follows.

\[
I_{x} = \iiint_{Q} (y^2 + z^2) \rho(x, y, z) \; dV \quad \text{Moment of inertia about x-axis}
\]

\[
I_{y} = \iiint_{Q} (x^2 + z^2) \rho(x, y, z) \; dV \quad \text{Moment of inertia about y-axis}
\]

\[
I_{z} = \iiint_{Q} (x^2 + y^2) \rho(x, y, z) \; dV \quad \text{Moment of inertia about z-axis}
\]

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

\[
I_{x} = I_{xz} + I_{xy} \quad I_{y} = I_{yz} + I_{xy} \quad \text{and} \quad I_{z} = I_{xz} + I_{xz}
\]

where $I_{x}$, $I_{y}$, and $I_{z}$ are as follows.

\[
I_{xy} = \iiint_{Q} z^2 \rho(x, y, z) \; dV
\]

\[
I_{xz} = \iiint_{Q} y^2 \rho(x, y, z) \; dV
\]

\[
I_{yz} = \iiint_{Q} x^2 \rho(x, y, z) \; dV
\]
**NOTE** In engineering and physics, the moment of inertia of a mass is used to find the time required for the mass to reach a given speed of rotation about an axis, as shown in Figure 14.60. The greater the moment of inertia, the longer a force must be applied for the mass to reach the given speed.

*Figure 14.60*
Try It 5

Find the mass of the solid bounded by the graphs of $2x + 3y + 6z = 12$, $x = 0$, $y = 0$, and $z = 0$, given that the density is $\rho(x, y, z) = ky$.

$$\iiint 1 \, dV$$

$$2x - 3y - 2z = 12$$

$$\frac{z}{2} = \frac{12 - 7y - 2x}{6}$$

$$\int_{0}^{y} \int_{0}^{\frac{12-3y-2x}{6}} \int_{0}^{\frac{2x+3y}{2}} (ky) \, dz \, dx \, dy$$
14.6 Exercises

In Exercises 1–8, evaluate the iterated integral.

1. \[ \int_0^1 \int_0^y (x + y + z) \, dx \, dz \, dy \]
2. \[ \int_0^1 \int_0^{e^{-z}} e^{-yz} \, dx \, dy \]
3. \[ \int_0^1 \int_0^y x \, dz \, dy \]
4. \[ \int_0^1 \int_0^{x^2} z \, dz \, dx \]
5. \[ \int_0^1 \int_0^y 2xe^{-x^2} \, dy \, dz \]
6. \[ \int_0^1 \int_0^{\cos z} \ln z \, dy \, dz \]
7. \[ \int_0^1 \int_0^{1-x} x \cos y \, dz \, dy \]
8. \[ \int_0^1 \int_0^{\sin y} \sin y \, dz \, dx \]

In Exercises 9 and 10, use a computer algebra system to evaluate the iterated integral.

9. \[ \int_0^1 \int_0^{\sqrt{y + x}} y \, dz \, dy \]
10. \[ \int_0^1 \int_0^{x+y-z} y \, dz \, dy \]

In Exercises 11 and 12, use a computer algebra system to approximate the iterated integral.

11. \[ \int_0^1 \int_0^{\sin^{-1} x} z^2 \, dy \, dx \]
12. \[ \int_0^1 \int_0^{z^2} ze^{-x^2} \, dy \, dx \]

In Exercises 13–18, set up a triple integral for the volume of the solid.

13. The solid in the first octant bounded by the coordinate planes and the plane \( z = 5 - x - y \)
14. The solid bounded by \( z = 9 - x^2, z = 0, y = 0 \), and \( y = 2x \)
15. The solid bounded by \( z = 6 - x^2 - y^2 \) and \( z = 0 \)
16. The solid bounded by \( z = \sqrt{10 - x^2 - y^2} \) and \( z = 0 \)
17. The solid that is the common interior below the sphere \( x^2 + y^2 + z^2 = 80 \) and above the paraboloid \( z = \frac{1}{2}(x^2 + y^2) \)
18. The solid bounded above by the cylinder \( z = 4 - x^2 \) and below by the paraboloid \( z = x^2 + 3y^2 \)

Volume In Exercises 19–22, use a triple integral to find the volume of the solid shown in the figure.

19.

20.

21. \[ z = 36 - x^2 - y^2 \]

22. \[ z = 0 \]

Volume In Exercises 23–26, use a triple integral to find the volume of the solid bounded by the graphs of the equations.

23. \( z = 4 - x^2, y = 4 - x^2 \), first octant
24. \( z = 9 - x^2, y = -x^2 + 2, y = 0, z = 0, x \geq 0 \)
25. \( z = 2 - y, z = 4 - y^2, x = 0, x = 3, y = 0 \)
26. \( z = x, y = x + 2, y = x^2 \), first octant

In Exercises 27–32, sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

27. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dy \, dz \, dx \).
28. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dx \, dz \, dy \).
29. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dz \, dy \, dx \).
30. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dz \, dx \, dy \).
31. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dz \, dy \, dx \).
32. \[ \int_0^1 \int_0^{\sqrt{z-x}} dz \, dy \]
Rewrite using the order \( dz \, dx \, dy \).

In Exercises 33–36, list the six possible orders of integration for the triple integral over the solid region \( Q \), \[ \iiint_Q xyz \, dV \].

33. \( Q = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq 3\} \)
34. \( Q = \{(x, y, z): 0 \leq x \leq 2, x^2 \leq y \leq 4, 0 \leq z \leq 2 - x\} \)
35. \( Q = \{(x, y, z): x^2 + y^2 \leq 9, 0 \leq z \leq 4\} \)
36. \( Q = \{(x, y, z): 0 \leq x \leq 1, y \leq 1 - x^2, 0 \leq z \leq 6\} \)
In Exercises 37 and 38, the figure shows the region of integration for the given integral. Rewrite the integral as an equivalent iterated integral in the five other orders.

37. \[ \int_0^1 \int_0^1 \int_0^1 \; dz \, dy \, dx \]
38. \[ \int_0^1 \int_0^1 \int_0^1 \; dz \, dy \, dx \]

Mass and Center of Mass In Exercises 39–42, find the mass and the indicated coordinates of the center of mass of the solid of given density bounded by the graphs of the equations.

39. Find \( \bar{x} \) using \( \rho(x, y, z) = k \). 
   \( Q: 2x + 3y + 5z = 12 \), \( x = 0 \), \( y = 0 \), \( z = 0 \)
40. Find \( \bar{y} \) using \( \rho(x, y, z) = ky \). 
   \( Q: 3x + 3y + 5z = 15 \), \( x = 0 \), \( y = 0 \), \( z = 0 \)
41. Find \( \bar{z} \) using \( \rho(x, y, z) = k \). 
   \( Q: z = 4 - x \), \( z = 0 \), \( y = 0 \), \( x = 0 \)
42. Find \( \bar{y} \) using \( \rho(x, y, z) = k \). 
   \( Q: \frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1 \) (a, b, c > 0), \( x = 0 \), \( y = 0 \), \( z = 0 \)

Mass and Center of Mass In Exercises 43 and 44, set up the triple integrals for finding the mass and the center of mass of the solid bounded by the graphs of the equations.

43. \( x = 0 \), \( x = b \), \( y = 0 \), \( y = b \), \( z = 0 \), \( z = b \)
   \( \rho(x, y, z) = kx \)
44. \( x = 0 \), \( x = a \), \( y = 0 \), \( y = b \), \( z = 0 \), \( z = c \)
   \( \rho(x, y, z) = kz \)

Think About It: The center of mass of a solid of constant density is shown in the figure. In Exercises 45–48, make a conjecture about how the center of mass \( \bar{r}, \bar{y}, \bar{z} \) will change for the nonconstant density \( \rho(x, y, z) \). Explain.

45. \( \rho(x, y, z) = kx \)
46. \( \rho(x, y, z) = ky \)
47. \( \rho(x, y, z) = k(y + 2) \)
48. \( \rho(x, y, z) = kyz(y + 2)^2 \)

Centroid In Exercises 49–54, find the centroid of the solid region bounded by the graphs of the equations or described by the figure. Use a computer algebra system to evaluate the triple integrals. (Assume uniform density and find the center of mass.)

49. \( z = \frac{h}{r} \sqrt{x^2 + y^2}, z = 0 \)
50. \( y = \sqrt{9 - x^2}, z = y, z = 0 \)
51. \( z = \sqrt{16 - x^2 - y^2}, z = 0 \)
52. \( z = \frac{1}{y^2 + 1}, z = 0, x = -2, x = 2, y = 0, y = 1 \)
53. \( \bar{z} = \frac{1}{k} \)
54. \( \bar{z} = \frac{1}{k} \)

Moments of Inertia In Exercises 55–58, find \( I_x, I_y, \) and \( I_z \) for the solid of given density. Use a computer algebra system to evaluate the triple integrals.

55. (a) \( \rho = k \)
   (b) \( \rho = kxyz \)
56. (a) \( \rho(x, y, z) = k \)
   (b) \( \rho(x, y, z) = k(x^2 + y^2) \)
57. (a) \( \rho(x, y, z) = k \)
   (b) \( \rho = ky \)
58. (a) \( \rho = kz \)
   (b) \( \rho = k(4 - z) \)

Moments of Inertia In Exercises 59 and 60, verify the moments of inertia for the solid of uniform density. Use a computer algebra system to evaluate the triple integrals.

59. \( I_x = \frac{1}{12}m(3a^2 + L^2) \)
   \( I_y = \frac{1}{12}ma^2 \)
   \( I_z = \frac{1}{12}m(3a^2 + L^2) \)
60. \( I_x = \frac{1}{2} \pi a (a^2 + b^2) \)
\( I_y = \frac{1}{2} \pi b (b^2 + c^2) \)
\( I_z = \frac{1}{2} \pi c (c^2 + a^2) \)

**CAPSTONE**

68. **Think About It**  Of the integrals (a)–(c), which one is equal to \( \int \int \int \int f(x, y, z) \, dx \, dy \, dz \)? Explain.

(a) \( \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dz \, dx \, dy \)
(b) \( \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dy \, dx \, dz \)
(c) \( \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz \)

**Average Value**  In Exercises 69–72, find the average value of the function over the given solid. The average value of a continuous function \( f(x, y, z) \) over a solid region \( Q \) is

\[ \frac{1}{V} \int \int \int f(x, y, z) \, dV \]

where \( V \) is the volume of the solid region \( Q \).

69. \( f(x, y, z) = z^2 + 4 \) over the cube in the first octant bounded by the coordinate planes and the planes \( x = 1, y = 1, z = 1 \)
70. \( f(x, y, z) = xyz \) over the cube in the first octant bounded by the coordinate planes and the planes \( x = 4, y = 4, z = 4 \)
71. \( f(x, y, z) = x + y + z \) over the tetrahedron in the first octant with vertices \((0, 0, 0), (2, 0, 0), (0, 2, 0)\) and \((0, 0, 2)\)
72. \( f(x, y, z) = x + y \) over the solid bounded by the sphere \( x^2 + y^2 + z^2 = 3 \)

67. Consider two solids, solid \( A \) and solid \( B \), of equal weight as shown below.

(a) Because the solids have the same weight, which has the greater density? 
(b) Which solid has the greater moment of inertia? Explain. 
(c) The solids are rolled down an inclined plane. They are started at the same time and at the same height. Which will reach the bottom first? Explain.

**PUTNAM EXAM CHALLENGE**

77. Evaluate

\[ \lim_{n \to \infty} \left( \int_0^1 \cdots \int_0^1 \cos \left( \frac{2 \pi}{n} x_1 + x_2 + \cdots + x_n \right) \, dx_1 \, dx_2 \cdots \, dx_n \right) \]

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