13.1 Introduction to Functions of Several Variables

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

So far in this text, you have dealt only with functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by a force $w = FD$ and the volume of a right circular cylinder $V = \pi r^2h$ are both functions of two variables. The volume of a rectangular solid $(V = lwh)$ is a function of three variables. The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

\[
\begin{align*}
  z &= f(x, y) = x^2 + xy \\
  &\quad \text{Function of two variables} \\
  &\quad \text{2 variables} \\
  y &= f(x, y, z) = x + 2y - 3z \\
  &\quad \text{Function of three variables} \\
  &\quad \text{3 variables}
\end{align*}
\]

EXPLORATION

Comparing Dimensions
Without using a graphing utility, describe the graph of each function of two variables.

- $a. \ z = x^2 + y^2$
- $b. \ z = x + y$
- $c. \ z = x^2 + y$
- $d. \ z = \sqrt{x^2 + y^2}$
- $e. \ z = \sqrt{1 - x^2 + y^2}$

DEFINITION OF A FUNCTION OF TWO VARIABLES

Let $D$ be a set of ordered pairs of real numbers. If to each ordered pair $(x, y)$ in $D$ there corresponds a unique real number $f(x, y)$, then $f$ is called a function of $x$ and $y$. The set $D$ is the domain of $f$, and the corresponding set of values for $f(x, y)$ is the range of $f$.

For the function given by $z = f(x, y)$, $x$ and $y$ are called the independent variables and $z$ is called the dependent variable.

Similar definitions can be given for functions of three, four, or $n$ variables, where the domains consist of ordered triples $(x_1, x_2, x_3)$, quadruples $(x_1, x_2, x_3, x_4)$, and $n$-tuples $(x_1, x_2, \ldots, x_n)$. In all cases, the range is a set of real numbers. In this chapter, you will study only functions of two or three variables.

As with functions of one variable, the most common way to describe a function of several variables is with an equation, and unless otherwise restricted, you can assume that the domain is the set of all points for which the equation is defined. For instance, the domain of the function given by

\[
f(x, y) = x^2 + y^2
\]

is assumed to be the entire $xy$-plane. Similarly, the domain of

\[
f(x, y) = \ln xy
\]

is the set of all points $(x, y)$ in the plane for which $xy > 0$. This consists of all points in the first and third quadrants.

\[
\begin{align*}
  \text{X \& Y Both Negative -3rd Quadrant} \\
  \text{X \& Y Both Positive -1st Quadrant}
\end{align*}
\]
Try It 1

Find the domain of the function.

\[ f(x, y) = \ln(4 - xy). \]

\[ \ln \text{ (positive number)} = \text{real value.} \]

\[ 4 - xy > 0 \]
\[ 4 > xy \]
\[ \frac{y}{x} > 1 \]
\[ y < \frac{y}{x} \] (4, 1)

Domain
\[ xy < 4 \]
\[ y = \frac{4}{x} \]
\[ 1 < y \]
Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

\[
\begin{align*}
(f \pm g)(x, y) &= f(x, y) \pm g(x, y) & \text{Sum or difference} \\
(fg)(x, y) &= f(x, y)g(x, y) & \text{Product} \\
\frac{f}{g}(x, y) &= \frac{f(x, y)}{g(x, y)} & \text{Quotient}
\end{align*}
\]

You cannot form the composite of two functions of several variables. However, if \( h \) is a function of several variables and \( g \) is a function of a single variable, you can form the **composite function** \((g \circ h)(x, y)\) as follows.

\[
(g \circ h)(x, y) = g(h(x, y))
\]

The domain of this composite function consists of all \((x, y)\) in the domain of \( h \) such that \( h(x, y) \) is in the domain of \( g \). For example, the function given by

\[
f(x, y) = \sqrt{16 - 4x^2 - y^2}
\]

can be viewed as the composite of the function of two variables given by \( h(x, y) = \frac{\sqrt{16 - 4x^2 - y^2}}{16 - 4x^2 - y^2} \) and the function of a single variable given by \( g(u) = \sqrt{u} \). The domain of this function is the set of all points lying on or inside the ellipse given by \( 4x^2 + y^2 = 16 \).

A function that can be written as a sum of functions of the form \( cx^m y^n \) (where \( c \) is a real number and \( m \) and \( n \) are nonnegative integers) is called a **polynomial function** of two variables. For instance, the functions given by

\[
\begin{align*}
f(x, y) &= x^3 + y^3 - 2xy + x + 2 \\
g(x, y) &= 3xy^2 + x - 2
\end{align*}
\]

are polynomial functions of two variables. A **rational function** is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.
The Graph of a Function of Two Variables

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The graph of a function $f$ of two variables is the set of all points $(x, y, z)$ for which $z = f(x, y)$ and $(x, y)$ is in the domain of $f$. This graph can be interpreted geometrically as a surface in space, as discussed in Sections 11.5 and 11.6. In Figure 13.2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the $xy$-plane is $D$, the domain of $f$. To each point $(x, y)$ in $D$ there corresponds a point $(x, y, z)$ on the surface, and, conversely, to each point $(x, y, z)$ on the surface there corresponds a point $(x, y)$ in $D$.

Try It 2

What is the range of $f(x, y) = \sqrt{x^2 + y^2}$? Describe the graph of $f$.

Surface: $z = \sqrt{x^2 + y^2}$

Domain:

$\sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} \geq 0$

Entire $xy$-Plane

Range

$[0, \infty)$

$z \geq 0$

Traces

$z = 0 \quad x^2 + y^2 = 0$

$z = 1 \quad x^2 + y^2 = 1$

$z = 4 \quad x^2 + y^2 = 4$

$z = f(x, y) = ___$

$z = f(x, y) = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

$x^2 + y^2 - z^2 = 0$

(Cone) $z \geq 0$
To sketch a surface in space by hand, it helps to use traces in planes parallel to the coordinate planes, as shown in Figure 13.3. For example, to find the trace of the surface in the plane $z = 2$, substitute $z = 2$ in the equation $z = \sqrt{16 - 4x^2 - y^2}$ and obtain

$$2 = \sqrt{16 - 4x^2 - y^2} \Rightarrow \frac{x^2}{3} + \frac{y^2}{12} = 1.$$

So, the trace is an ellipse centered at the point $(0, 0, 2)$ with major and minor axes of lengths $4\sqrt{3}$ and $2\sqrt{3}$.

Traces are also used with most three-dimensional graphing utilities. For instance, Figure 13.4 shows a computer-generated version of the surface given in Example 2. For this graph, the computer took 25 traces parallel to the $xy$-plane and 12 traces in vertical planes.

If you have access to a three-dimensional graphing utility, use it to graph several surfaces.
Level Curves

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar \( z = f(x, y) \) is assigned to the point \((x, y)\). A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of \( f(x, y) \) is constant. For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars**. In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms**, as shown in Figure 13.6. Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called **equipotential lines**.

![Figure 13.5](image1)

- Level curves show the lines of equal pressure (isobars) measured in millibars.

![Figure 13.6](image2)

- Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map**. For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

A contour map depicts the variation of \( z \) with respect to \( x \) and \( y \) by the spacing between level curves. Much space between level curves indicates that \( z \) is changing slowly, whereas little space indicates a rapid change in \( z \). Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose \( z \)-values that are evenly spaced.

![Figure 13.8](image3)
Try It 3
Ellipsoidal Paraboloid $z = x^2 + y^2$

The graph of $f(x, y) = x^2 + y^2$ is shown in Figure 1. Sketch a contour map for this surface using level curves corresponding to $c = 0, 2, 4, 6, 8$.

Figure 1

\[
\begin{align*}
C &= 0 & 0 &= x^2 + y^2 \\
C &= 2 & 2 &= x^2 + y^2 & \text{Circle } r = \sqrt{2} \\
C &= 4 & 4 &= x^2 + y^2 & \text{Circle } r = 2 \\
C &= 6 & 6 &= x^2 + y^2 & \text{Circle } r = \sqrt{6} \\
C &= 8 & 8 &= x^2 + y^2 & \text{Circle } r = \sqrt{8}
\end{align*}
\]
Try It 4

The surface given by $z = e^{xy}$ is shown in Figure 1. Sketch a contour map for this surface.

**f(x,y) = z = e^{xy}**

**Domain:** $e^{xy}$

**Entire Real Plane**

**Range:** $e^{xy}$

$e$ to any power $\geq 0$

$z > 0$ $(0, \infty)$

$C=1$

$1 = e^{xy}$

$\ln 1 = \ln e^{xy}$

$0 = xy$

$0 = xy$

Any point on the x or y axis

$C=2$

$2 = e^{xy}$

$\ln 2 = \ln e^{xy}$

$\ln 2 = xy$

$y = \frac{\ln 2}{x}$

$y = \frac{0.69}{x}$

$C=3$

$y = \frac{\ln 3}{x} = \frac{1.09}{x}$

$C=4$

$y = \frac{\ln 4}{x} = \frac{1.4}{x}$

$C=\frac{1}{2}$

$y = \frac{\ln \frac{1}{2}}{x} = -\frac{\ln 2}{x}$

$C=\frac{1}{4}$

$y = \frac{\ln \frac{1}{4}}{x} = -\frac{\ln 4}{x}$
One example of a function of two variables used in economics is the Cobb-Douglas production function. This function is used as a model to represent the numbers of units produced by varying amounts of labor and capital. If \( x \) measures the units of labor and \( y \) measures the units of capital, the number of units produced is given by:

\[
f(x, y) = Cx^a y^{1-a}
\]

where \( C \) and \( a \) are constants with \( 0 < a < 1 \).

**Try It 5**

A manufacturer estimates a production function to be \( f(x, y) = 50x^6y^{0.7} \), where \( x \) is the number of units of labor and \( y \) is the number of units of capital. Compare the production level when \( x = 750 \) and \( y = 1000 \) with the production level when \( x = 2250 \) and \( y = 3000 \).

\[
f(750, 1000) = 50(750)^6(1000)^{0.7} = 45,865.74
\]

\[
f(2250, 3000) = 50(2250)^6(3000)^{0.7} = 137,597
\]

\[
f(x, y) = 50x^6y^{0.7}
\]

\[
c = 140,000
\]

\[
140,000 = 50x^6y^{0.7}
\]

\[
\frac{140,000}{50x^6} = y^{0.7}
\]

\[
(\frac{2800}{x^6})^{\frac{10}{7}} = (y^{\frac{20}{7}})^{\frac{10}{7}}
\]

For \( c = 140,000 \):

\[
\frac{2800^{\frac{40}{7}}}{x^{\frac{30}{7}}} = y
\]
Level Surfaces \( w = f(x, y, z) \)

The concept of a level curve can be extended by one dimension to define a level surface. If \( f \) is a function of three variables and \( c \) is a constant, the graph of the equation \( f(x, y, z) = c \) is a level surface of the function \( f \), as shown in Figure 13.14.

With computers, engineers and scientists have developed other ways to view functions of three variables. For instance, Figure 13.15 shows a computer simulation that uses color to represent the temperature distribution of fluid inside a pipe fitting.

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Try It 6

Describe the level surfaces of the function

\[
f(x, y, z) = w = x^2 + y^2 - z
\]

- Level surfaces: \( x^2 + y^2 - z = c \)
- \( c = 0 \)
- \( c = 4 \)

- \( D = x^2 + y^2 - z \)
- Paraboloid vertex at \( (0, 0, 0) \)
- \( z = x^2 + y^2 \)

- \( C = 4 \)
- \( y = x^2 + y^2 - z \)
- \( z = x^2 + y^2 - 4 \)
- Shift Paraboloid 4 units Down

- \( C = 4 \)
- \( -y = x^2 + y^2 - z \)
- \( z = x^2 + y^2 + 4 \)
- Shift Paraboloid 4 units Up
13.1 Exercises

In Exercises 1 and 2, use the graph to determine whether $z$ is a function of $x$ and $y$. Explain.

1.

In Exercises 3–6, determine whether $z$ is a function of $x$ and $y$.

3. $x^2 + 3y^2 - xy = 10$  
4. $x^2 + 2xy - y^2 = 4$

5. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$  
6. $z + x \ln y - 8z = 0$

In Exercises 7–18, find and simplify the function values.

7. $f(x, y) = xy$
   (a) $(3, 2)$  
   (b) $(-1, 4)$  
   (c) $(30, 5)$  
   (d) $(5, y)$  
   (e) $(x, 2)$  
   (f) $(5, l)$

8. $f(x, y) = 4 - x^2 - 4y^2$
   (a) $(0, 0)$  
   (b) $(0, 1)$  
   (c) $(2, 3)$  
   (d) $(1, x)$  
   (e) $(x, 0)$  
   (f) $(l, 1)$

9. $f(x, y) = xe^y$
   (a) $(5, 0)$  
   (b) $(3, 2)$  
   (c) $(2, -1)$  
   (d) $(5, y)$  
   (e) $(x, 2)$  
   (f) $(l, l)$

10. $g(x, y) = \ln|x + y|$
    (a) $(1, 0)$  
    (b) $(0, -1)$  
    (c) $(0, e)$  
    (d) $(1, 1)$  
    (e) $(e, e/2)$  
    (f) $(2, 5)$

11. $h(x, y, z) = \frac{xy}{z}$
    (a) $(2, 3, 9)$  
    (b) $(1, 0, 1)$  
    (c) $(-2, 3, 4)$  
    (d) $(5, 4, -6)$

12. $f(x, y, z) = \sqrt{x + y + z}$
    (a) $(0, 5, 4)$  
    (b) $(6, 8, -3)$  
    (c) $(4, 6, 2)$  
    (d) $(10, -4, -3)$

13. $f(x, y) = x \sin y$
    (a) $(2, \pi/4)$  
    (b) $(3, 1)$  
    (c) $(-3, \pi/3)$  
    (d) $(4, \pi/2)$

14. $V(r, h) = \pi r^2 h$
    (a) $(3, 10)$  
    (b) $(5, 2)$  
    (c) $(4, 8)$  
    (d) $(6, 4)$

15. $g(x, y) = \int_{2t}^{3t} dt$
    (a) $(4, 0)$  
    (b) $(4, 1)$  
    (c) $(4, 2)$  
    (d) $(4, 3)$

16. $g(x, y) = \int\frac{1}{t} dt$
    (a) $(4, 1)$  
    (b) $(6, 3)$  
    (c) $(2, 5)$  
    (d) $(4, 7)$

17. $f(x, y) = 2x + ye^y$
    (a) $f(x + \Delta x, y) - f(x, y) = f(x, y)\frac{\Delta x}{\Delta y}$
    (b) $f(x, y + \Delta y) - f(x, y) = f(x, y)\frac{\Delta y}{\Delta x}$

18. $f(x, y) = 3x^2 - 2y$
    (a) $f(x + \Delta x, y) - f(x, y) = f(x, y)\frac{\Delta x}{\Delta y}$
    (b) $f(x, y + \Delta y) - f(x, y) = f(x, y)\frac{\Delta y}{\Delta x}$

In Exercises 19–30, describe the domain and range of the function.

19. $f(x, y) = x^2 + y^2$
20. $f(x, y) = e^{xy}$

21. $g(x, y) = \sqrt{x + y}$
22. $g(x, y) = \frac{\sqrt{x}}{\sqrt{y}}$

23. $z = \frac{x + y}{xy}$
24. $z = \frac{x + y}{x - y}$

25. $f(x, y) = \sqrt[4]{x^2 + y^2}$
26. $f(x, y) = \sqrt[4]{4 - x^2 - y^2}$

27. $f(x, y) = \arccos(x + y)$
28. $f(x, y) = \arcsin(y/x)$

29. $f(x, y) = \ln(4 - x - y)$
30. $f(x, y) = \ln(xy - 0)$

31. Think About It The graphs labeled (a), (b), (c), and (d) are graphs of the function $f(x, y) = -4x/(x^2 + y^2 + 1)$. Match the four graphs with the points in space from which the surface is viewed. The four points are $(20, 15, 25)$, $(-15, 10, 20)$, $(20, 0, 0)$, and $(20, 0, 0)$.
32. **Think About It** Use the function given in Exercise 31.
   (a) Find the domain and range of the function.
   (b) Identify the points in the xy-plane at which the function value is 0.
   (c) Does the surface pass through all the octants of the rectangular coordinate system? Give reasons for your answer.

In Exercises 33–40, sketch the surface given by the function.

33. \( f(x, y) = 4 \)  
34. \( f(x, y) = 6 - 2x - 3y \)  
35. \( f(x, y) = y^2 \)  
36. \( g(x, y) = \frac{1}{2}y \)  
37. \( z = -x^2 - y^3 \)  
38. \( z = \frac{1}{2}\sqrt{x^2 + y^2} \)  
39. \( f(x, y) = e^{-x} \)  
40. \( f(x, y) = \begin{cases} y, & x \geq 0, y \geq 0 \\ 0, & x < 0 \text{ or } y < 0 \end{cases} \)

In Exercises 41–44, use a computer algebra system to graph the function.

41. \( z = y^2 - x^2 + 1 \)  
42. \( z = \frac{1}{4}\sqrt{444 - 16x^2 - 9y^2} \)  
43. \( f(x, y) = -x^2 - 3y^2 \)  
44. \( f(x, y) = x \sin y \)

In Exercises 45–48, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a), (b), (c), and (d).]

45. \( f(x, y) = e^{-x^2 - y^2} \)  
46. \( f(x, y) = e^{-(x^2 + y^2)} \)

47. \( f(x, y) = \ln|y - x^2| \)  
48. \( f(x, y) = \cos\left(\frac{x^2 + 2xy}{4}\right) \)

In Exercises 49–56, describe the level curves of the function. Sketch the level curves for the given c-values.

49. \( z = x + y, \quad c = -1, 0, 2, 4 \)  
50. \( z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10 \)  
51. \( z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4 \)  
52. \( f(x, y) = \sqrt{9 - x^2 - y^2}, \quad c = 0, 1, 2, 3 \)  
53. \( f(x, y) = xy, \quad c = \pm 1, \pm 2, \ldots, \pm 6 \)  
54. \( f(x, y) = e^{0.3xy}, \quad c = 2, 3, 4, 5, 6 \)  
55. \( f(x, y) = \sqrt{x^2 + y^2}, \quad c = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 7 \)  
56. \( f(x, y) = \ln|x - y|, \quad c = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2 \)

In Exercises 57–60, use a graphing utility to graph six level curves of the function.

57. \( f(x, y) = x^2 - y^2 + 2 \)  
58. \( f(x, y) = |xy| \)  
59. \( g(x, y) = \frac{8}{1 + x^2 + y^2} \)  
60. \( h(x, y) = 3 \sin|x| + |y| \)

**Writing About Concepts**

61. What is a graph of a function of two variables? How is it interpreted geometrically? Describe level curves.

62. All of the level curves of the surface given by \( z = f(x, y) \) are concentric circles. Does this imply that the graph of \( f \) is a hemisphere? Illustrate your answer with an example.

63. Construct a function whose level curves are lines passing through the origin.

**Capstone**

64. Consider the function \( f(x, y) = xy \), for \( x \geq 0 \) and \( y \geq 0 \).
   (a) Sketch the graph of the surface given by \( f \).
   (b) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = f(x, y) - 3 \). Explain your reasoning.
   (c) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = -f(x, y) \). Explain your reasoning.
   (d) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = \frac{1}{2}f(x, y) \). Explain your reasoning.
   (e) On the surface in part (a), sketch the graph of \( z = f(x, y) \).
Writing In Exercises 65 and 66, use the graphs of the level curves (c-values evenly spaced) of the function \( f \) to write a description of a possible graph of \( f \). Is the graph of \( f \) unique? Explain.

65. [Graph A]

66. [Graph B]

67. Investment In 2009, an investment of $1000 was made in a bond earning 6% compounded annually. Assume that the buyer pays tax at rate \( R \) and the annual rate of inflation is \( I \). In the year 2019, the value \( V \) of the investment in constant 2009 dollars is

\[
V(f, R) = \frac{1000}{1 + 0.06(1 - R)}^{10}. \]

Use this function of two variables to complete the table.

<table>
<thead>
<tr>
<th>Tax Rate</th>
<th>Inflation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.03</td>
</tr>
<tr>
<td>0.28</td>
<td>0.35</td>
</tr>
</tbody>
</table>

68. Investment A principal of $5000 is deposited in a savings account that earns interest at a rate of \( r \) (written as a decimal), compounded continuously. The amount \( M(r, t) \) after \( t \) years is \( M(r, t) = 5000e^{rt} \). Use this function of two variables to complete the table.

<table>
<thead>
<tr>
<th>Number of Years</th>
<th>Rate</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
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</tr>
<tr>
<td>15</td>
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</tr>
<tr>
<td>20</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

In Exercises 69–74, sketch the graph of the level surface \( f(x, y, z) = c \) at the given value of \( c \).

69. \( f(x, y, z) = x - y + z \), \( c = 1 \)

70. \( f(x, y, z) = 4x + y + 2z \), \( c = 4 \)

71. \( f(x, y, z) = x^2 + y^2 + z^2 \), \( c = 9 \)

72. \( f(x, y, z) = x^2 + \frac{1}{2}y^2 - z \), \( c = 1 \)

73. \( f(x, y, z) = 4x^2 + 4y^2 - z^2 \), \( c = 0 \)

74. \( f(x, y, z) = \sin x - z \), \( c = 0 \)

75. Forestry The Doyle Log Rule is one of several methods used to determine the lumber yield of a log (in board-feet) in terms of its diameter \( d \) (in inches) and its length \( L \) (in feet). The number of board-feet is

\[
N(d, L) = \left( \frac{d - 4}{4} \right)^{10} L. \]

(a) Find the number of board-feet of lumber in a log 22 inches in diameter and 12 feet in length.

(b) Find \( N(30, 12) \).

76. Queuing Model The average length of time that a customer waits in line for service is

\[
W(x, y) = \frac{1}{x - y}, \quad x > y
\]

where \( y \) is the average arrival rate, written as the number of customers per unit of time, and \( x \) is the average service rate, written in the same units. Evaluate each of the following.

(a) \( W(15, 9) \) (b) \( W(15, 13) \) (c) \( W(12, 7) \) (d) \( W(5, 2) \)

77. Temperature Distribution The temperature \( T \) (in degrees Celsius) at any point \( (x, y) \) in a circular steel plate of radius 10 meters is \( T = 600 - 0.75x^2 - 0.75y^2 \), where \( x \) and \( y \) are measured in meters. Sketch some of the isothermals curves.

78. Electric Potential The electric potential \( V \) at any point \( (x, y) \) is

\[
V(x, y) = \frac{5}{\sqrt{x^2 + y^2}}.
\]

Sketch the equipotential curves for \( V = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \) and \( V = \frac{7}{4} \).

79. Cobb-Douglas Production Function Use the Cobb-Douglas production function (see Example 5) to show that if the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

80. Cobb-Douglas Production Function Show that the Cobb-Douglas production function \( z = Cx^a y^b \) can be rewritten as

\[
\ln z = \ln C + a \ln x + b \ln y.
\]

81. Construction Cost A rectangular box with an open top has a length of \( x \) feet, a width of \( y \) feet, and a height of \( z \) feet. It costs \$1.20 per square foot to build the base and \$0.75 per square foot to build the sides. Write the cost \( C \) of constructing the box as a function of \( x, y, \) and \( z \).

82. Volume A propane tank is constructed by welding hemispheres to the ends of a right circular cylinder. Write the volume \( V \) of the tank as a function of \( r \) and \( L \), where \( r \) is the radius of the cylinder and hemispheres, and \( L \) is the length of the cylinder.

83. Ideal Gas Law According to the Ideal Gas Law, \( PV = kT \), where \( P \) is pressure, \( V \) is volume, \( T \) is temperature (in Kelvin), and \( k \) is a constant of proportionality. A tank contains 2000 cubic inches of nitrogen at a pressure of 28 pounds per square inch and a temperature of 300 K.

(a) Determine \( k \).

(b) Write \( P \) as a function of \( V \) and \( T \) and describe the level curves.
84. **Modeling Data** The table shows the net sales $x$ (in billions of dollars), the total assets $y$ (in billions of dollars), and the shareholder’s equity $z$ (in billions of dollars) for Wal-Mart for the years 2002 through 2007. (Source: 2007 Annual Report for Wal-Mart)

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>201.2</td>
<td>226.5</td>
<td>252.8</td>
<td>281.5</td>
<td>206.9</td>
<td>345.0</td>
</tr>
<tr>
<td>$y$</td>
<td>97.3</td>
<td>90.2</td>
<td>102.5</td>
<td>117.1</td>
<td>135.6</td>
<td>151.2</td>
</tr>
<tr>
<td>$z$</td>
<td>35.2</td>
<td>39.5</td>
<td>43.6</td>
<td>49.4</td>
<td>53.2</td>
<td>61.6</td>
</tr>
</tbody>
</table>

A model for these data is

$$z = f(x, y) = 0.026x + 0.316y + 5.04.$$  

(a) Use a graphing utility and the model to approximate $z$ for the given values of $x$ and $y$.

(b) Which of the two variables in this model has the greater influence on shareholder’s equity?

(c) Simplify the expression for $f(x, 95)$ and interpret its meaning in the context of the problem.

85. **Meteorology** Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points $A$, $B$, and $C$ with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

86. **Acid Rain** The acidity of rainwater is measured in units called pH. A pH of 7 is neutral, smaller values are increasingly acidic, and larger values are increasingly alkaline. The map shows curves of equal pH and gives evidence that downwind of heavily industrialized areas the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.

87. **Atmosphere** The contour map shown in the figure was computer generated using data collected by satellite instrumentation. Color is used to show the “ozone hole” in Earth’s atmosphere. The purple and blue areas represent the lowest levels of ozone and the green areas represent the highest levels. (Source: National Aeronautics and Space Administration)

Figure for 87

(a) Do the level curves correspond to equally spaced ozone levels? Explain.

(b) Describe how to obtain a more detailed contour map.

88. **Geology** The contour map in the figure represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. (Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Tenth Edition)

Figure for 86

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain.

**True or False?** In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. If $f(x_0, y_0) = f(x_1, y_1)$, then $x_0 = x_1$ and $y_0 = y_1$.

90. If $f$ is a function, then $f(ax, ay) = a^2 f(x, y)$.

91. A vertical line can intersect the graph of $z = f(x, y)$ at most once.

92. Two different level curves of the graph of $z = f(x, y)$ can intersect.