13.2 Limits and Continuity

- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

**Neighborhoods in the Plane**

In this section, you will study limits and continuity involving functions of two or three variables. The section begins with functions of two variables. At the end of the section, the concepts are extended to functions of three variables.

We begin our discussion of the limit of a function of two variables by defining a two-dimensional analog to an interval on the real number line. Using the formula for the distance between two points \((x, y)\) and \((x_0, y_0)\) in the plane, you can define the \(\delta\)-neighborhood about \((x_0, y_0)\) to be the disk centered at \((x_0, y_0)\) with radius \(\delta > 0\)

\[
\{(x, y): \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}
\]

as shown in Figure 13.18. When this formula contains the less than inequality sign, \(<\), the disk is called open, and when it contains the less than or equal to inequality sign, \(\le\), the disk is called closed. This corresponds to the use of \(<\) and \(\le\) to define open and closed intervals.

![An open disk](Figure 13.18)

![The boundary and interior points of a region R](Figure 13.19)

A point \((x_0, y_0)\) in a plane region \(R\) is an interior point of \(R\) if there exists a \(\delta\)-neighborhood about \((x_0, y_0)\) that lies entirely in \(R\), as shown in Figure 13.19. If every point in \(R\) is an interior point, then \(R\) is an open region. A point \((x_0, y_0)\) is a boundary point of \(R\) if every open disk centered at \((x_0, y_0)\) contains points inside \(R\) and points outside \(R\). By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is closed. A region that contains some but not all of its boundary points is neither open nor closed.
Limit of a Function of Two Variables

**DEFINITION OF THE LIMIT OF A FUNCTION OF TWO VARIABLES**

Let $f$ be a function of two variables defined, except possibly at $(x_0, y_0)$, on an open disk centered at $(x_0, y_0)$, and let $L$ be a real number. Then

$$\lim_{(x,y) \to (x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

**NOTE** Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius $\delta$, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 13.20.

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference. To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left. If the function approaches the same limit from the right and from the left, you can conclude that the limit exists. However, for a function of two variables, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point $(x, y)$ is allowed to approach $(x_0, y_0)$ from any direction. If the value of

$$\lim_{(x,y) \to (x_0, y_0)} f(x, y)$$

is not the same for all possible approaches, or paths, to $(x_0, y_0)$, the limit does not exist.
Try It 1

Show that

\[ \lim_{(x,y) \to (2,3)} y = 3. \]

For \( E > 0 \), there exists a \( S \) Neighborhood About \((2,3)\) such that

\[ \left| f(x,y) - L \right| = \left| y - 3 \right| < \varepsilon \]

when

\[ 0 < \sqrt{(x-2)^2 + (y-3)^2} < S \]

\[ \left| y - 3 \right| = \sqrt{(y-3)^2} \leq \sqrt{(x-2)^2 + (y-3)^2} < S \]

Let \( \varepsilon = S \)

If \( 0 < \sqrt{(x-2)^2 + (y-3)^2} < S \)

\[ 0 < \sqrt{(y-3)^2} < S \]

\[ 0 < |y-3| < S \]

Then \( 0 < |y-3| < \varepsilon \)

If \( S = \varepsilon \)
Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables. (See Theorem 1.2 in Section 1.3.) Some of these properties are used in the next example.

Try It 2

Evaluate

\[ \lim_{{(x, y) \to (\pi/4, 2)}} y \sin (xy). \]

Direct Substitution

\[ = 2 \sin \left( \left( \frac{\pi}{4} \right) \left( \frac{\pi}{2} \right) \right) \]

\[ = 2 \sin \left( \frac{\pi^2}{8} \right) \]

\[ = 2 \left( \frac{\pi}{2} \right) \]

\[ = (2) \]

\[ z = f \left( \frac{\pi}{4}, 2 \right) = 2 \]

Point in 3 Space

\[ \left( \frac{\pi}{4}, 2, 2 \right) \]
Try It 3

Evaluate

$$\lim_{(x, y) \to (0, 0)} \frac{3xy^4}{x + y}$$

Solution

In this case, the limits of the numerator and of the denominator are both 0, and so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from the graph of $f$ shown in the figure, it seems reasonable that the limit might be 0.

So, you can try applying the definition to $L = 0$. First note that

$$y^4 \leq (x + y)^4 \text{ and } \frac{x}{x + y} \leq 1.$$

Then, in a $\delta$-neighborhood about $(0, 0)$, you have $0 < (x + 4)^4 < \delta$, and it follows that for $(x, y) \neq (0, 0)$,

$$|f(x, y) - 0| = \left| \frac{3xy^4}{x + y} \right| = 3|y^4| \left| \frac{x}{x + y} \right| \leq 3y^4 \leq 3(x + y)^4 < 3\delta.$$

So, you can choose $\delta = \varepsilon/3$ and conclude that

$$\lim_{(x, y) \to (0, 0)} \frac{3xy^4}{x + y} = 0.$$
**Example 3 Verifying a Limit**

Evaluate \( \lim_{{(x,y) \to (0,0)}} \frac{5x^2 y}{x^2 + y^2} = \frac{0}{0} \) \( \text{In determine} \)

**Solution** In this case, the limits of the numerator and of the denominator are both 0, and so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from the graph of \( f \) in Figure 13.21, it seems reasonable that the limit might be 0. So, you can try applying the definition to \( L = 0 \). First, note that

\[
|y| \leq \sqrt{x^2 + y^2} \quad \text{and} \quad \frac{x^2}{x^2 + y^2} \leq 1.
\]

Then, in a \( \delta \)-neighborhood about \((0, 0)\), you have \( 0 < \sqrt{x^2 + y^2} < \delta \), and it follows that, for \((x, y) \neq (0, 0)\),

\[
|f(x, y) - 0| = \left| \frac{5x^2 y}{x^2 + y^2} \right| < \varepsilon.
\]

If \( 0 < \sqrt{x^2 + y^2} < \delta \)

So, you can choose \( \delta = \varepsilon/5 \) and conclude that

\[
\lim_{{(x,y) \to (0,0)}} \frac{5x^2 y}{x^2 + y^2} = 0.
\]

\[\text{Fig. 13.21} \quad \text{Surface:} \quad f(x, y) = \frac{5x^2 y}{x^2 + y^2} \]
For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit
\[
\lim_{{(x, y) \to (0, 0)}} \frac{1}{{x^2 + y^2}} = \infty
\]
does not exist because the values of \( f(x, y) \) increase without bound as \((x, y)\) approaches \((0, 0)\) along any path (see Figure 13.22).

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function approaches different values along different paths.

\[
\lim_{{(x, y) \to (0, 0)}} \frac{1}{{x^2 + y^2}} \text{ does not exist.}
\]
Try It 4

Show that the following limit does not exist.

\[
\lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + 2y^6} = \frac{0}{0}
\]

Indeterminate

Domain: All points in \(xy\) plane \((0,0)\)

Along \(x = y^3\):

\[
(-y^3, y) \quad \text{Limit is} \quad -\frac{1}{3}
\]

Along \(x = -y^3\):

\[
\lim_{(-y^3, y) \to (0,0)} \frac{(-y^3)y^3}{(-y^3)^2 + 2y^6} = \frac{-y^6}{y^6 + 2y^6} = \frac{-y^6}{3y^6}
\]

\[
\lim_{(-y^3, y) \to (0,0)} \frac{-y^6}{y^6 + 2y^6} = -\frac{1}{3}
\]

2nd Path

\[
(\frac{y}{3}, y) \to (0,0) \quad \frac{y^6}{y^6 + 2y^6} = \frac{y^6}{3y^6}
\]

\[
\lim_{(\frac{y}{3}, y) \to (0,0)} \frac{y^6}{y^6 + 2y^6} = +\frac{1}{3}
\]

Limit Does Not Exist
Continuity of a Function of Two Variables

Notice in Example 2 that the limit of \( f(x, y) = \frac{5x^2y}{x^2 + y^2} \) as \( (x, y) \to (1, 2) \) can be evaluated by direct substitution. That is, the limit is \( f(1, 2) = 2 \). In such cases the function \( f \) is said to be **continuous** at the point \( (1, 2) \).

**DEFINITION OF CONTINUITY OF A FUNCTION OF TWO VARIABLES**

A function \( f \) of two variables is **continuous at a point** \( (x_0, y_0) \) in an open region \( R \) if \( f(x_0, y_0) \) is equal to the limit of \( f(x, y) \) as \( (x, y) \) approaches \( (x_0, y_0) \). That is,

\[
\lim_{{(x, y) \to (x_0, y_0)}} f(x, y) = f(x_0, y_0).
\]

The function \( f \) is **continuous in the open region** \( R \) if it is continuous at every point in \( R \).

In Example 3, it was shown that the function

\[
f(x, y) = \frac{5x^2y}{x^2 + y^2}
\]

is not continuous at \((0, 0)\). However, because the limit at this point exists, you can remove the discontinuity by defining \( f \) at \((0, 0)\) as being equal to its limit there. Such a discontinuity is called **removable**. In Example 4, the function

\[
f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2
\]

was also shown not to be continuous at \((0, 0)\), but this discontinuity is **nonremovable**.
THEOREM 13.1 CONTINUOUS FUNCTIONS OF TWO VARIABLES

If $k$ is a real number and $f$ and $g$ are continuous at $(x_0, y_0)$, then the following functions are continuous at $(x_0, y_0)$.

1. Scalar multiple: $kf$
2. Sum and difference: $f \pm g$
3. Product: $fg$
4. Quotient: $f/g$, if $g(x_0, y_0) \neq 0$

Theorem 13.1 establishes the continuity of polynomial and rational functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables. For instance, the functions whose graphs are shown in Figures 13.24 and 13.25 are continuous at every point in the plane.

The function $f$ is continuous at every point in the plane. Figure 13.24

The function $f$ is continuous at every point in the plane. Figure 13.25
The next theorem states conditions under which a composite function is continuous.

**THEOREM 13.2 CONTINUITY OF A COMPOSITE FUNCTION**

If \( h \) is continuous at \((x_0, y_0)\) and \( g \) is continuous at \( h(x_0, y_0) \), then the composite function given by \((g \circ h)(x, y) = g(h(x, y))\) is continuous at \((x_0, y_0)\). That is,

\[
\lim_{(x, y) \to (x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).
\]

**NOTE** Note in Theorem 13.2 that \( h \) is a function of two variables and \( g \) is a function of one variable.

**Try It 5**

Discuss the continuity of the function,

\[
f(x, y) = \frac{xy}{(x - y)^2}
\]

Surface:

\[
f(x, y) = \frac{xy}{(x + y)^2}
\]

\[(x-y)^2 \neq 0\]

\[x-y \neq 0\]

\[x \neq y\]

Not Continuous Along

The line \( y = x \)
Continuity of a Function of Three Variables

The preceding definitions of limits and continuity can be extended to functions of three variables by considering points \((x, y, z)\) within the open sphere

\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2.
\]

The radius of this sphere is \(\delta\), and the sphere is centered at \((x_0, y_0, z_0)\), as shown in Figure 13.28. A point \((x_0, y_0, z_0)\) in a region \(R\) in space is an interior point of \(R\) if there exists a \(\delta\)-sphere about \((x_0, y_0, z_0)\) that lies entirely in \(R\). If every point in \(R\) is an interior point, then \(R\) is called open.

**DEFINITION OF CONTINUITY OF A FUNCTION OF THREE VARIABLES**

A function \(f\) of three variables is **continuous at a point** \((x_0, y_0, z_0)\) in an open region \(R\) if \(f(x_0, y_0, z_0)\) is defined and is equal to the limit of \(f(x, y, z)\) as \((x, y, z)\) approaches \((x_0, y_0, z_0)\). That is,

\[
\lim_{{(x, y, z) \to (x_0, y_0, z_0)}} f(x, y, z) = f(x_0, y_0, z_0).
\]

The function \(f\) is **continuous in the open region** \(R\) if it is continuous at every point in \(R\).

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**Try It 6**

Test the continuity of the function

\[
f(x, y, z) = \frac{z}{x^2 + y^2 - 4}.
\]

\[x^2 + y^2 - 4 \neq 0\]

Not Continuous

\[x^2 + y^2 \neq 4\]

Cylinder of radius = 2
a) \[ f(x, y) = \frac{y^4}{x^2 - y^2} \]
\[ x^2 - y^2 \neq 0 \]
\[ y^2 \neq x^2 \]
\[ \gamma = \pm x \]

Not Continuous at \( \gamma = x \)

\( \gamma = -x \)

b) \[ g(x, y) = \frac{xy}{e^{xy}} \]
\[ e^{xy} \geq 1 \]

\[ \frac{(1)(-2)}{e^{-2}} = \frac{-2}{e^{-1}} = \frac{-1}{e^0} \]

Always Continuous

\[ z = -2e \]
In Exercises 1–4, use the definition of the limit of a function of two variables to verify the limit.

1. \( \lim_{(x, y) \to (1, 3)} \frac{x}{y} = x = 1 \)  
2. \( \lim_{(x, y) \to (-1, -1)} \frac{x}{y} = x = 4 \)  
3. \( \lim_{(x, y) \to (1, -1)} \frac{x}{y} = y = -3 \)  
4. \( \lim_{(x, y) \to (1, 1)} \frac{x}{y} = y = 6 \)

In Exercises 5–8, find the indicated limit by using the limits:

5. \( \lim_{(x, y) \to (a, b)} \frac{f(x, y) - g(x, y)}{f(x, y) + g(x, y)} \)  
6. \( \lim_{(x, y) \to (a, b)} \frac{f(x, y)}{g(x, y)} \)  
7. \( \lim_{(x, y) \to (a, b)} [f(x, y)g(x, y)] \)  
8. \( \lim_{(x, y) \to (a, b)} \frac{f(x, y) + g(x, y)}{f(x, y)} \)

In Exercises 9–22, find the limit and discuss the continuity of the function.

9. \( \lim_{(x, y) \to (a, b)} (2x^2 + y) \)  
10. \( \lim_{(x, y) \to (0, 0)} (x + 4y + 1) \)  
11. \( \lim_{(x, y) \to (0, 0)} e^{x+y} \)  
12. \( \lim_{(x, y) \to (0, 0)} \frac{x+y}{x^2+y} \)  
13. \( \lim_{(x, y) \to (0, 0)} \frac{x}{x^2+y^2} \)  
14. \( \lim_{(x, y) \to (1, 2)} \frac{x+y}{x^2+y^2} \)  
15. \( \lim_{(x, y) \to (1, 1)} \frac{xy}{x+y} \)  
16. \( \lim_{(x, y) \to (1, 1)} \frac{x}{\sqrt{x+y}} \)  
17. \( \lim_{(x, y) \to (0, 0)} \arcsin xy \)  
18. \( \lim_{(x, y) \to (0, 0)} \frac{\sin x}{x} \)  
19. \( \lim_{(x, y) \to (0, 0)} \frac{\arccos (x/y)}{x^2+y^2} \)  
20. \( \lim_{(x, y) \to (0, 0)} \frac{\arccos (x/y)}{1+xy} \)  
21. \( \lim_{(x, y) \to (0, 0)} \frac{x^2}{xy} \)  
22. \( \lim_{(x, y) \to (0, 0)} xe^{xy} \)
In Exercises 23–36, find the limit (if it exists). If the limit does not exist, explain why.

23. \( \lim_{{(x, y) \to (0, 0)}} \frac{xy - 1}{1 + xy} \)
24. \( \lim_{{(x, y) \to (1, -1)}} \frac{x^2y}{1 + 2xy} \)
25. \( \lim_{{(x, y) \to (0, 0)}} \frac{x^2 - y^2}{x - y} \)
26. \( \lim_{{(x, y) \to (0, 0)}} \frac{1}{x^2 + y^2} \)
27. \( \lim_{{(x, y) \to (1, 2)}} \frac{x^2 - y^2}{x - y} \)
28. \( \lim_{{(x, y) \to (0, 0)}} \frac{x^4 - 4y^4}{x^2 + 2y^2} \)
29. \( \lim_{{(x, y) \to (0, 0)}} \frac{x - y}{\sqrt{x} - \sqrt{y}} \)
30. \( \lim_{{(x, y) \to (2, 1)}} \frac{x - y - 1}{\sqrt{x} - y - 1} \)
31. \( \lim_{{(x, y) \to (0, 0)}} \frac{x + y}{x^2 + y^2} \)
32. \( \lim_{{(x, y) \to (0, 0)}} \frac{x^2}{x^2 - y^2} \)
33. \( \lim_{{(x, y) \to (0, 0)}} (x^2 + 1)(y^2 + 1) \)
34. \( \lim_{{(x, y) \to (0, 0)}} \ln(x^2 + y^2) \)

In Exercises 37 and 38, discuss the continuity of the function and evaluate the limit of \( f(x, y) \) (if it exists) as \( (x, y) \to (0, 0) \).

37. \( f(x, y) = e^{xy} \)

38. \( f(x, y) = 1 - \frac{\cos(x^2 + y^2)}{x^2 + y^2} \)

In Exercises 39–42, use a graphing utility to make a table showing the values of \( f(x, y) \) at the given points for each path. Use the result to make a conjecture about the limit of \( f(x, y) \) as \( (x, y) \to (0, 0) \). Determine whether the limit exists analytically and discuss the continuity of the function.

39. \( f(x, y) = \frac{xy}{x^2 + y^2} \)
   Path: \( y = 0 \)
   Points: \((1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)\)
   Path: \( y = x \)
   Points: \((1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)\)

40. \( f(x, y) = \frac{y}{x^2 + y^2} \)
   Path: \( y = 0 \)
   Points: \((1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)\)
   Path: \( y = x \)
   Points: \((1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)\)

41. \( f(x, y) = -\frac{y^2}{x^2 + y^4} \)
   Path: \( x = y^2 \)
   Points: \((1, 1), (0.25, 0.5), (0.01, 0.1), (0.0001, 0.01), (0.000001, 0.001)\)
   Path: \( x = -y^2 \)
   Points: \((-1, 1), (-0.25, 0.5), (-0.01, 0.1), (-0.0001, 0.01), (-0.000001, 0.001)\)
18. \( f(x, y) = 3x^2 - 2y \)

(a) \( \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \)

(b) \( \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \)

\[ \frac{[3(x+\Delta x)^2 - 2y] - [3x^2 - 2y]}{\Delta x} = \]

\[ \text{Answer} \]

18 a) \( 6x + 3\Delta x \)

18 b) \( -\frac{2\Delta y}{\Delta y} = -2 \)
42. \( f(x, y) = \frac{2x - y^2}{2x^2 + y} \)

Path: \( y = 0 \)
Points: \((1, 0), (0.25, 0), (0.01, 0), (0.0001, 0)\)
Path: \( y = x \)
Points: \((1, 1), (0.25, 0.25), (0.01, 0.01), (0.0001, 0.0001)\)

In Exercises 43–46, discuss the continuity of the functions \( f \) and \( g \). Explain any differences.

43. \( f(x, y) = \begin{cases} 
\frac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases} \)

\( g(x, y) = \begin{cases} 
\frac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
1, & (x, y) = (0, 0)
\end{cases} \)

In Exercises 47–52, use a computer algebra system to graph the function and find \( \lim_{(x, y) \to (0, 0)} f(x, y) \) (if it exists).

47. \( f(x, y) = \sin x + \sin y \)
48. \( f(x, y) = \sin \frac{1}{x} + \cos \frac{1}{y} \)
49. \( f(x, y) = \frac{x^2 y}{x^2 + 2y} \)
50. \( f(x, y) = \frac{x^2 + y^2}{x^2 y} \)
51. \( f(x, y) = \frac{5xy}{x^2 + 2y^2} \)
52. \( f(x, y) = \frac{6xy}{x^2 + y^2 + 1} \)

In Exercises 53–58, use polar coordinates to find the limit. [Hint: Let \( x = r \cos \theta \) and \( y = r \sin \theta \), and note that \((x, y) \to (0, 0)\) implies \( r \to 0 \).]

53. \( \lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2} \)
54. \( \lim_{(x, y) \to (0, 0)} \frac{x^4 + y^4}{x^2 + y^2} \)
55. \( \lim_{(x, y) \to (0, 0)} \frac{x^5 y^2}{x^2 + y^2} \)
56. \( \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \)
57. \( \lim_{(x, y) \to (0, 0)} \cos(x^4 + y^2) \)
58. \( \lim_{(x, y) \to (0, 0)} \sin(\sqrt{x^2 + y^2}) \)

In Exercises 59–62, use polar coordinates and L'Hôpital's Rule to find the limit.

59. \( \lim_{(x, y) \to (0, 0)} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \)
60. \( \lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \)
61. \( \lim_{(x, y) \to (0, 0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2} \)
62. \( \lim_{(x, y) \to (0, 0)} \frac{(x^2 + y^2)\ln(x^2 + y^2)}{x^2 + y^2} \)

In Exercises 63–68, discuss the continuity of the function.

63. \( f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \)
64. \( f(x, y, z) = \frac{z}{x^2 + y^2 - 4} \)
65. \( f(x, y, z) = \frac{\sin z}{x^2 + y^2} \)
66. \( f(x, y, z) = xy \sin z \)

67. \( f(x, y) = \begin{cases} 
sin xy, & x \neq 0, y \\
1, & x = 0 \end{cases} \)
68. \( f(x, y) = \begin{cases} 
\frac{\sin(x^2 - y^2)}{x^2 - y^2}, & x \neq y \\
1, & x = y \end{cases} \)

In Exercises 69–72, discuss the continuity of the composite function \( f \circ g \).

69. \( f(t) = t^2 \)
70. \( f(t) = \frac{1}{t} \)
71. \( f(t) = \frac{1}{t} \)
72. \( f(t) = \frac{1}{t - 1} \)

In Exercises 73–78, find each limit.

(a) \( \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \)
(b) \( \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \)
73. \( f(x, y) = x^2 - 4y \)
74. \( f(x, y) = x^2 + y^2 \)
75. \( f(x, y) = \frac{x}{y} \)
76. \( f(x, y) = \frac{1}{x + y} \)
77. \( f(x, y) = 3x + xy - 2y \)
78. \( f(x, y) = \sqrt{x + y} \)
True or False? In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

79. If \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \), then \( \lim_{x \to 0} f(x, 0) = 0 \).

80. If \( \lim_{(x, y) \to (0, 0)} f(0, y) = 0 \), then \( \lim_{y \to 0} f(x, y) = 0 \).

81. If \( f \) is continuous for all nonzero \( x \) and \( y \), and \( f(0, 0) = 0 \), then \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \).

82. If \( g \) and \( h \) are continuous functions of \( x \) and \( y \), and \( f(x, y) = g(x) + h(y) \), then \( f \) is continuous.

83. Consider \( \lim_{(x, y) \to (0, 0)} \frac{x^2 + y^2}{xy} \) (see figure).

84. Consider \( \lim_{(x, y) \to (0, 0)} \frac{xy}{x^3 + y^3} \) (see figure).

86. \( \lim_{(x, y) \to (0, 0)} \tan^{-1} \left( \frac{1}{x^2 + y^2 + z^2} \right) \)

87. Find the following limit.
\[
\lim_{(x, y) \to (0, 1)} \tan^{-1} \frac{x^3 + 1}{x^2 + (y - 1)^2}
\]

88. For the function
\[
f(x, y) = x^2 - \frac{y^2}{x^2 + y^2}
\]
define \( f(0, 0) \) such that \( f \) is continuous at the origin.

89. Prove that
\[
\lim_{(x, y) \to (a, b)} \left[ f(x, y) + g(x, y) \right] = L_1 + L_2
\]
where \( f(x, y) \) approaches \( L_1 \) and \( g(x, y) \) approaches \( L_2 \) as \( (x, y) \to (a, b) \).

90. Prove that if \( f \) is continuous and \( f(a, b) < 0 \), there exists a \( \delta \)-neighborhood about \( (a, b) \) such that \( f(x, y) < 0 \) for every point \( (x, y) \) in the neighborhood.

**WRITING ABOUT CONCEPTS**

91. Define the limit of a function of two variables. Describe a method for showing that
\[
\lim_{(x, y) \to (a, b)} f(x, y) = L
\]
and for using.

92. State the definition of continuity of a function of two variables.

93. Determine whether each of the following statements is true or false. Explain your reasoning.
   (a) If \( \lim_{(x, y) \to (0, 0)} f(x, y) = 4 \), then \( \lim_{x \to 0} f(x, 0) = 4 \).
   (b) If \( \lim_{x \to 0} f(x, 3) = 4 \), then \( \lim_{x \to 0} f(x, y) = 4 \).
   (c) If \( \lim_{(x, y) \to (0, 0)} f(x, 3) = 4 \), then \( \lim_{(x, y) \to (0, 0)} f(x, y) = 4 \).
   (d) If \( \lim_{(x, y) \to (0, 0)} f(3, y) = 0 \), then for any real number \( k \), \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \).

**CAPSTONE**

94. (a) If \( f(2, 3) = 4 \), can you conclude anything about \( \lim_{(x, y) \to (2, 3)} f(x, y) \)? Give reasons for your answer.
   (b) If \( \lim_{(x, y) \to (2, 3)} f(x, y) = 4 \), can you conclude anything about \( f(2, 3) \)? Give reasons for your answer.

In Exercises 85 and 86, use spherical coordinates to find the limit. [Hint: Let \( x = \rho \cos \phi \cos \theta \), \( y = \rho \sin \phi \sin \theta \), and \( z = \rho \cos \phi \), and note that \( (x, y, z) \to (0, 0, 0) \) implies \( \rho \to 0^+ \).]

85. \( \lim_{(x, y, z) \to (0, 0, 0)} x^2 + y^2 + z^2 \)