13.8 Extrema of Functions of Two Variables

- Find absolute and relative extrema of a function of two variables.
- Use the Second Partials Test to find relative extrema of a function of two variables.

Absolute Extrema and Relative Extrema

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 13.15 below, the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function $f$ of two variables, defined on a closed bounded region $R$. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \leq f(x, y) \leq f(c, d)$$

for all $(x, y)$ in $R$ are called the minimum and maximum of $f$ in the region $R$, as shown in Figure 13.64. Recall from Section 13.2 that a region in the plane is closed if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and bounded. A region in the plane is called bounded if it is a subregion of a closed disk in the plane.

**THEOREM 13.15 EXTREME VALUE THEOREM**

Let $f$ be a continuous function of two variables $x$ and $y$ defined on a closed bounded region $R$ in the $xy$-plane.

1. There is at least one point in $R$ at which $f$ takes on a minimum value.
2. There is at least one point in $R$ at which $f$ takes on a maximum value.

**DEFINITION OF RELATIVE EXTREMA**

Let $f$ be a function defined on a region $R$ containing $(x_0, y_0)$.

1. The function $f$ has a relative minimum at $(x_0, y_0)$ if

$$f(x, y) \geq f(x_0, y_0)$$

for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

2. The function $f$ has a relative maximum at $(x_0, y_0)$ if

$$f(x, y) \leq f(x_0, y_0)$$

for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

To say that $f$ has a relative maximum at $(x_0, y_0)$ means that the point $(x_0, y_0, z_0)$ is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, $f$ has a relative minimum at $(x_0, y_0)$ if $(x_0, y_0, z_0)$ is at least as low as all nearby points on the graph. (See Figure 13.65.)
To locate relative extrema of $f$, you can investigate the points at which the gradient of $f$ is $0$ or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of $f$.

**DEFINITION OF CRITICAL POINT**

Let $f$ be defined on an open region $R$ containing $(x_0, y_0)$. The point $(x_0, y_0)$ is a critical point of $f$ if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 13.11 that if $f$ is differentiable and

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)i + f_y(x_0, y_0)j$$

$$= 0i + 0j$$

then every directional derivative at $(x_0, y_0)$ must be 0. This implies that the function has a horizontal tangent plane at the point $(x_0, y_0)$, as shown in Figure 13.66. It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 13.16.

**THEOREM 13.16 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS**

If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ is a critical point of $f$.

**EXPLORATION**

Use a graphing utility to graph

$$z = x^3 - 3xy + y^3$$

using the bounds $0 \leq x \leq 3$, $0 \leq y \leq 3$, and $-3 \leq z \leq 3$. This view makes it appear as though the surface has an absolute minimum. But does it?
Try It 1

Determine the relative extrema of \( f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13 \).

\[ f_x(x, y) = -2x + 8 \]
\[ -2x + 8 = 0 \]
\[ -2x = -8 \]
\[ x = 4 \]

\[ f_y(x, y) = -10y - 10 \]
\[ -10y - 10 = 0 \]
\[ -10y = 10 \]
\[ y = -1 \]

Critical Point \((4, -1)\)

\[ -x^2 + 8x - 5y^2 - 10y - 13 \]

\[ -1 (x^2 - 8x + 16) - 5(y^2 + 2y + 1) - 13 + 16 + 5 \]

\[ -1(x - 4)^2 - 5(y + 1)^2 + 8 \]

Max of This Expression Is 8 when \( x = 4 \) and \( y = -1 \)

Absolute Maximum

Vertex \((4, -1, 8)\)
Example 1 shows a relative minimum occurring at one type of critical point—the type for which both \( f_x(x, y) \) and \( f_y(x, y) \) are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either \( f_x(x, y) \) or \( f_y(x, y) \) does not exist.

**Try It 2**

Determine the relative extrema of \( f(x, y) = \sqrt{x^2 + y^2} + 1 \).

\[
\begin{align*}
    f_x &= \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2x) \\
    f_x &= \frac{x}{\sqrt{x^2 + y^2}} \\
    f_y &= \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2y) \\
    f_y &= \frac{y}{\sqrt{x^2 + y^2}}
\end{align*}
\]

Partial Derivatives Exist and Are Not Zero For All Points In The Plane Except They Don't Exist At \((0, 0)\) Critical Point

\[
\begin{align*}
    f(x, y) &= \sqrt{x^2 + y^2} + 1 > 1 \\
    f(0, 0) &= 1 \quad \Rightarrow \quad (0, 0) \text{ Absolute Minimum}
\end{align*}
\]
The Second Partial Test

Theorem 13.16 tells you that to find relative extrema you need only examine values of \( f(x, y) \) at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield saddle points, which are neither relative maxima nor relative minima.

As an example of a critical point that does not yield a relative extremum, consider the surface given by
\[
f(x, y) = y^2 - x^2
\]

as shown in Figure 13.69. At the point \((0, 0)\), both partial derivatives are 0. The function \( f \) does not, however, have a relative extremum at this point because in any open disk centered at \((0, 0)\) the function takes on both negative values (along the \( x \)-axis) and positive values (along the \( y \)-axis). So, the point \((0, 0, 0)\) is a saddle point of the surface. (The term “saddle point” comes from the fact that surfaces such as the one shown in Figure 13.69 resemble saddles.)

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema, because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partial Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is best left to a course in advanced calculus.

**THEOREM 13.17 SECOND PARTIALS TEST**

Let \( f \) have continuous second partial derivatives on an open region containing a point \((a, b)\) for which
\[
f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.
\]

To test for relative extrema of \( f \), consider the quantity
\[
d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.
\]

1. If \( d > 0 \) and \( f_{xx}(a, b) > 0 \), then \( f \) has a relative minimum at \((a, b)\).
2. If \( d > 0 \) and \( f_{xx}(a, b) < 0 \), then \( f \) has a relative maximum at \((a, b)\).
3. If \( d < 0 \), then \((a, b, f(a, b))\) is a saddle point.
4. The test is inconclusive if \( d = 0 \).

**NOTE:** If \( d > 0 \), then \( f_{xx}(a, b) \) and \( f_{yy}(a, b) \) must have the same sign. This means that \( f_{xx}(a, b) \) can be replaced by \( f_{yy}(a, b) \) in the first two parts of the test.

A convenient device for remembering the formula for \( d \) in the Second Partial Test is given by the \( 2 \times 2 \) determinant:
\[
d = \begin{vmatrix}
    f_{xx}(a, b) & f_{xy}(a, b) \\
    f_{xy}(a, b) & f_{yy}(a, b)
\end{vmatrix}
\]

where \( f_{xy}(a, b) = f_{yx}(a, b) \) by Theorem 13.3.
Try It 3

Find the relative extrema of \( f(x, y) = 4xy - x^4 - y^4 \).

Relative maximum
(1, 1, 2)

Relative maximum
(-1, -1, 2)

Saddle point:
(0, 0, 0)

\[ f(x, y) = 4xy - x^4 - y^4 \]

**2nd Partial Test**

\[ f_{xx} = -12x^2 \]

\[ f_{yy} = -12y^2 \]

\[ f_{xy} = 4 \]

\[ (0,0) \quad d = f_{xx}(0,0)f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 \]

\[ d = (0)(0) - (-16) = -16 \]

\[ d = -16 \quad d < 0 \quad \text{Saddle Point} \]

\[ (1,0) \quad d = f_{xx}(1,0)f_{yy}(1,0) - \left[f_{xy}(1,0)\right]^2 \]

\[ (1)(1) - (-4)^2 = 16 - 16 = 0 \quad \text{Only Real Critical Pts} \]

\[ (-1,0) \quad d = f_{xx}(-1,0)f_{yy}(-1,0) - \left[f_{xy}(-1,0)\right]^2 \]

\[ (-1)(-4) - (-4)^2 = 16 - 16 = 0 \quad \text{Critical Pts} \]

\[ (0,1) \quad d = f_{xx}(0,1)f_{yy}(0,1) - \left[f_{xy}(0,1)\right]^2 \]

\[ (1)(1) - (1)^2 = 1 \quad \text{Relative Maximum} \]

\[ (1,1) \quad d = f_{xx}(1,1)f_{yy}(1,1) - \left[f_{xy}(1,1)\right]^2 \]

\[ (1)(1) - (2)^2 = -3 \quad \text{Relative Maximum} \]

\[ (1,1,2) \quad d = f_{xx}(1,1)f_{yy}(1,1) - \left[f_{xy}(1,1)\right]^2 \]

\[ (1)(1) - (2)^2 = -3 \quad \text{Relative Maximum} \]
The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

\[ d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \neq 0 \]

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

**Try It 4**

Find the relative extrema of \( f(x, y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7 \).

\[
\begin{align*}
  f_x &= 3x^2 - 6x + 3 \\
  f_y &= 3y^2 + 12y + 12 \\
  3x^2 - 6x + 3 &= 0 \\
  3(y^2 + 4y + 4) &= 0 \\
  3(x - 1)(x - 1) &= 0 \\
  3(y + 2)(y + 2) &= 0 \\
  x &= 1 \\
  y &= -2
\end{align*}
\]

**Critical point** \((1, -2)\)

\(d = f_{xx}(1, -2) \cdot f_{yy}(1, -2) - [f_{xy}(1, -2)]^2 = 0\)

**2nd Partial Test Inconclusive**

\[
\begin{align*}
  (1.2, -2.2, -0.08) \\
  (-0.8, -2, -0.08)
\end{align*}
\]

**Saddle point**

\[
\begin{align*}
  (1.1, -2.1, -0.08)
\end{align*}
\]

**Critical point**

\[
\begin{align*}
  (1, -2, -)
\end{align*}
\]

**Saddle point**
Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, \( f(-2, 3) \) is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain. This is illustrated in Example 5.

**Try It 5**

Find the absolute extrema of the function
\[
f(x, y) = x^2 - 4xy
\]
on the closed region given by \( 0 \leq x \leq 4 \) and \( 0 \leq y \leq \sqrt{x} \).

**Boundary Case**

Along \( y = 0 \)
\[
f(x, 0) = x^2
\]
\[
f'_x = 2x
\]
\[
2x = 0 \quad x = 0
\]
\[
(0, 0)
\]
---

Along \( x = t \)
\[
f(t, y) = y^2 - 4ty\)
\[
= 16 - 16y
\]
\[
f'_y = -16
\]
---

Along \( y = \sqrt{x} \)
\[
f(x, \sqrt{x}) = x^2 - 4x(x^{1/2})
\]
\[
f'_x = 2x - 6x^{1/2} = 0
\]
Points To Check
For Max & Min.
\[
(0,0) \quad (4,0) \quad (4,-\frac{3}{2})
\]
---

Absolute Max
Absolute Min
(0,0) \quad (4,-\frac{3}{2})

Not \( \partial f / \partial x \) \text{ at } (4,0)
In Exercises 1–6, identify any extrema of the function by recognizing its given form or its form after completing the square. Verify your results by using the partial derivatives to locate any critical points and test for relative extrema.

1. \( g(x, y) = (x - 1)^2 + (y - 3)^2 \)
2. \( g(x, y) = 5 - (x - 3)^2 - (y + 2)^2 \)
3. \( f(x, y) = \sqrt{x^2 + y^2} + 1 \)
4. \( f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \)
5. \( f(x, y) = x^2 + y^2 + 2x - 6y + 6 \)
6. \( f(x, y) = -x^2 - y^2 + 10x + 12y - 64 \)

In Exercises 7–16, examine the function for relative extrema.

7. \( f(x, y) = 3x^2 + 2y^2 - 6x - 4y + 16 \)
8. \( f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5 \)
9. \( f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28 \)
10. \( f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3 \)
11. \( z = x^2 + xy + \frac{1}{2}y^2 - 2x + y \)
12. \( z = -5x^2 + 4xy - y^2 + 16x + 10 \)
13. \( f(x, y) = \sqrt{x^2 + y^2} \)
14. \( h(x, y) = (x^2 + y^2)^{1/3} + 1 \)
15. \( g(x, y) = 4 - |x| - |y| \)
16. \( f(x, y) = |x + y| - 2 \)

In Exercises 17–20, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.

17. \( z = \frac{-4x}{x^2 + y^2 + 1} \)
18. \( f(x, y) = y^2 - 3xy^2 - 3y^2 - 3x^2 + 1 \)
19. \( z = (x^2 + 4y)e^{x^2 - y^2} \)
20. \( z = e^{xy} \)

In Exercises 21–28, examine the function for relative extrema and saddle points.

21. \( h(x, y) = 80x + 80y - x^2 - y^2 \)
22. \( g(x, y) = x^2 - y^2 - x - y \)
23. \( g(x, y) = xy \)
24. \( h(x, y) = x^2 - 3xy - y^2 \)
25. \( f(x, y) = x^2 - xy - y^2 - 3x - y \)

26. \( f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1 \)
27. \( z = e^{-x} \sin y \)
28. \( z = \left(\frac{1}{2} - x^2 + y^2\right)e^{x^2 - y^2} \)

CAS: In Exercises 29 and 30, examine the function for extrema without using the derivative tests, and use a computer algebra system to graph the surface. (Hint: By observation, determine if it is possible for \( z \) to be negative. When is \( z \) equal to 0?)

29. \( z = \frac{(x - y)^4}{x^2 + y^2} \)
30. \( z = \frac{(x^2 - y^2)^2}{x^2 + y^2} \)

Think About It: In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function \( f(x, y) \) at the critical point \((x_0, y_0)\).

31. \( f_x(x_0, y_0) = 9, \ f_y(x_0, y_0) = 4, \ f_{xx}(x_0, y_0) = 6 \)
32. \( f_x(x_0, y_0) = -3, \ f_y(x_0, y_0) = -8, \ f_{xx}(x_0, y_0) = 2 \)
33. \( f_x(x_0, y_0) = -9, \ f_y(x_0, y_0) = 6, \ f_{xx}(x_0, y_0) = 10 \)
34. \( f_x(x_0, y_0) = 25, \ f_y(x_0, y_0) = 8, \ f_{xx}(x_0, y_0) = 10 \)
55. The function has continuous partial derivatives on an open region containing the critical point (3, 7). The function has a maximum at (3, 7), and d > 0 for the Second Partial Test. Determine the interval for f(3, 7) if f_x(3, 7) = 2 and f_y(3, 7) = 8.

56. A function f has continuous partial derivatives on an open region containing the critical point (a, b). If f_x(a, b) and f_y(a, b) have opposite signs, what is implied? Explain.

In Exercises 37–42, (a) find the critical points, (b) test for relative extrema, (c) list the critical points for which the Second Partial Test fails, and (d) use a computer algebra system to graph the function, labeling any extrema and saddle points.

37. f(x, y) = x^3 + y^3
38. f(x, y) = x^3 + y^3 - 6x^2 + 9y^3 + 12x + 27y^2 + 19
39. f(x, y) = (x - 1)^2 + (y + 3)^2
40. f(x, y) = \sqrt{x - 1} + (y + 2)^2
41. f(x, y) = x^{1/2} + y^{1/3}
42. f(x, y) = (x^2 + y^2)^{1/3}

In Exercises 43 and 44, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.

43. f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2
44. f(x, y, z) = 9 - [x(y - 1)(z + 2)]^3

In Exercises 45–54, find the absolute extrema of the function over the region R. (In each case, R contains the boundaries.) Use a computer algebra system to confirm your results.

45. f(x, y) = x^3 - 4xy + 5
   R = \{(x, y) : 1 \leq x \leq 4, 0 \leq y \leq 2\}
46. f(x, y) = x^3 + xy, \quad R = \{(x, y) : |x| \leq 2, |y| \leq 1\}
47. f(x, y) = 12 - 3x - 2y
   R: The triangular region in the xy-plane with vertices (2, 0), (0, 1), and (1, 2)
48. f(x, y) = 2x - y^2
   R: The triangular region in the xy-plane with vertices (2, 0), (0, 1), and (1, 2)
49. f(x, y) = 3x^2 + 2y^2 - 4y
   R: The region in the xy-plane bounded by the graphs of y = x^2 and y = 4
50. f(x, y) = 2x - 2xy + y^2
   R: The region in the xy-plane bounded by the graphs of y = x^2 and y = 1
51. f(x, y) = x^2 + 2xy + y^2
   R = \{(x, y) : |x| \leq 2, |y| \leq 1\}
52. f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y) : x^2 + y^2 \leq 8\}
53. f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}
   R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}
54. f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}
   R = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}

57. f(x, y) > 0 and f_y(x, y) < 0 for all (x, y).
58. All of the first and second partial derivatives of f are 0.
59. f_x(0, 0) = 0, f_y(0, 0) = 0

In Exercises 57–59, sketch the graph of an arbitrary function f satisfying the given conditions. State whether the function has any extrema or saddle points. (There are many correct answers.)

57. f(x, y) > 0 and f_y(x, y) < 0 for all (x, y).
58. All of the first and second partial derivatives of f are 0.
59. f_x(0, 0) = 0, f_y(0, 0) = 0

60. Consider the functions
   f(x, y) = x^2 - y^2 and g(x, y) = x^2 + y^2.
   (a) Show that both functions have a critical point at (0, 0).
   (b) Explain how f and g behave differently at this critical point.

True or False? In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

61. If f has a relative maximum at (x_0, y_0, z_0), then f_x(x_0, y_0) = 0.
62. If f(x_0, y_0) = f_y(x_0, y_0) = 0, then f has a relative maximum at (x_0, y_0, z_0).
63. Between any two relative minima of f, there must be at least one relative maximum of f.
64. If f is continuous for all x and y and has two relative minima, then f must have at least one relative maximum.
46 \quad f(x,y) = x^2 + xy

**Find Critical Points.**

\[ f_x(x,y) = 2x + y = 0 \]
\[ f_y(x,y) = x = 0 \quad \Rightarrow \quad x = 0 \]

\((0,0)\) Critical Point

2nd Partial Test
\[ f_{xx} = 2 \]
\[ f_{xy} = 0 \]
\[ f_{yx} = 1 \]
\[ d = f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \quad \text{Saddle Point!} \]

Along \( y = 1 \)
\[ f(x,1) = x^2 + x \]
\[ f'(x,1) = 2x + 1 = 0 \quad x = -\frac{1}{2}, \ y = 1 \]

Along \( y = -1 \)
\[ \text{You Do This One} \]

Along \( x = 2 \)
\[ f(2,y) = 4 + 2y = 0 \quad (2,-1,-) \]
\[ f'_y = 2 \quad \text{No Critical Values} \]

Along \( x = -2 \)
\[ \text{You Complete This One} \]

\((2,1,-)\)

\((-2,1,-)\)