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14.1 Iterated Integrals and Area in the Plane

Iterated Integrals

In Chapter 13, you saw that it is meaningful to differentiate functions of several variables with respect to one variable while holding the other variables constant. You can integrate functions of several variables by a similar procedure. For example, if you are given the partial derivative

\[ f_x(x, y) = 2xy \]

then, by considering \( y \) constant, you can integrate with respect to \( x \) to obtain

\[
\begin{align*}
  f(x, y) &= \int f_x(x, y) \, dx \\
  &= \int 2xy \, dx \\
  &= y \int 2x \, dx \\
  &= y(x^2) + C(y) \\
  &= x^2y + C(y).
\end{align*}
\]

The “constant” of integration, \( C(y) \), is a function of \( y \). In other words, by integrating with respect to \( x \), you are able to recover \( f(x, y) \) only partially. The total recovery of a function of \( x \) and \( y \) from its partial derivatives is a topic you will study in Chapter 15. For now, we are more concerned with extending definite integrals to functions of several variables. For instance, by considering \( y \) constant, you can apply the Fundamental Theorem of Calculus to evaluate

\[
\text{\( x \) is the variable of integration and \( y \) is fixed.}
\]

Replace \( x \) by the limits of integration, and the result is a function of \( y \).

Similarly, you can integrate with respect to \( y \) by holding \( x \) fixed. Both procedures are summarized as follows.

\[
\begin{align*}
  \int_{h_1(y)}^{h_2(y)} f_x(x, y) \, dx &= f(x, y) \bigg|_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \quad \text{With respect to } x \\
  \int_{g_1(x)}^{g_2(x)} f_y(x, y) \, dy &= f(x, y) \bigg|_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \quad \text{With respect to } y
\end{align*}
\]

Note that the variable of integration cannot appear in either limit of integration. For instance, it makes no sense to write

\[ \int_0^y x \, dx. \]
Try It 1

Evaluate $\int_{x^2}^{\sqrt{x}} (x^2 + y^2) \, dy$.

\[
\begin{align*}
\int_{x^2}^{\sqrt{x}} (x^2 + y^2) \, dy &= S x^2 \, dy + S y^2 \, dy \\
&= \left[ x^2 y + \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}} \\
&= \left( x^2 (\sqrt{x}) + \frac{(\sqrt{x})^3}{3} \right) - \left( x^2 (x^2) + \frac{(x^2)^3}{3} \right) \\
&= x^2 \sqrt{x} + \frac{1}{3}x^{\frac{3}{2}} - x^4 - \frac{x^6}{3}
\end{align*}
\]
Try It 2

Evaluate \( \int_0^1 \left[ \int_y^{2y} \left( 1 + 2x^2 + 2y^2 \right) dx \right] dy \).

\[
\int_0^1 \left( 1 + 2x^2 + 2y^2 \right) dx \\
= \left[ x + \frac{2x^3}{3} + 2y^2 x \right]_y^{2y} \\
= \left[ (2y) + \frac{2(2y)^3}{3} + 2y^2 (2y) \right] - \left[ (y) + \frac{2y^3}{3} + 2y^2 y \right] \\
= 2y + \frac{16y^3}{3} + 4y^3 - y - \frac{2y^3}{3} - 2y^3 \\
= \frac{2y}{3} + y - \frac{8y^3}{3} \\
\int_0^1 \left( \frac{2y}{3} + y \right) dy \\
= \left[ \frac{2y^4}{3} + \frac{y^2}{2} \right]_0^1 \\
= \frac{5y^4}{3} + \frac{y^2}{2} \\
= \left( \frac{5}{3} + \frac{1}{2} \right) - 0 \\
= \frac{10}{6} + \frac{3}{6} = \frac{12}{6}
\]
The integral in Example 2 is an **iterated integral**. The brackets used in Example 2 are normally not written. Instead, iterated integrals are usually written simply as
\[
\int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx
\]
and
\[
\int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
\]
The **inside limits of integration** can be variable with respect to the outer variable of integration. However, the **outside limits of integration** must be constant with respect to both variables of integration. After performing the inside integration, you obtain a "standard" definite integral, and the second integration produces a real number. The limits of integration for an iterated integral identify two sets of boundary intervals for the variables. For instance, in Example 2, the outside limits indicate that \( x \) lies in the interval \( 1 \leq x \leq 2 \) and the inside limits indicate that \( y \) lies in the interval \( 1 \leq y \leq x \). Together, these two intervals determine the **region of integration** \( R \) of the iterated integral, as shown in Figure 14.1.

Because an iterated integral is just a special type of definite integral—one in which the integrand is also an integral—you can use the properties of definite integrals to evaluate iterated integrals.

![Region of Integration](image)

The region of integration for
\[
\int_{1}^{2} \int_{1}^{x} f(x, y) \, dy \, dx
\]
**Figure 14.1**
Area of a Plane Region

In the remainder of this section, you will take a new look at an old problem—that of finding the area of a plane region. Consider the plane region \( R \) bounded by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), as shown in Figure 14.2. The area of \( R \) is given by the definite integral

\[
\int_{a}^{b} \left[ f_1(x) - f_2(x) \right] \, dx.
\]

Using the Fundamental Theorem of Calculus, you can rewrite the integrand \( g_2(x) - g_1(x) \) as a definite integral. Specifically, if you consider \( x \) to be fixed and let \( y \) vary from \( g_1(x) \) to \( g_2(x) \), you can write

\[
\int_{a}^{b} g_1(x) \, dx = \int_{a}^{b} g_2(x) \, dx = \int_{a}^{b} [g_2(x) - g_1(x)] \, dx.
\]

Combining these two integrals, you can write the area of the region \( R \) as an iterated integral

\[
\int_{a}^{b} \int_{g_1(x)}^{g_2(x)} \, dy \, dx = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} \, dx \, dy = \int_{a}^{b} [g_2(x) - g_1(x)] \, dx.
\]

Placing a representative rectangle in the region \( R \) helps determine both the order and the limits of integration. A vertical rectangle implies the order \( dy \, dx \), with the inside limits corresponding to the upper and lower bounds of the rectangle, as shown in Figure 14.2. This type of region is called vertically simple, because the outside limits of integration represent the vertical lines \( x = a \) and \( x = b \).

Similarly, a horizontal rectangle implies the order \( dx \, dy \), with the inside limits determined by the left and right bounds of the rectangle, as shown in Figure 14.3. This type of region is called horizontally simple, because the outside limits represent the horizontal lines \( y = c \) and \( y = d \). The iterated integrals used for these two types of simple regions are summarized as follows.

### AREA OF A REGION IN THE PLANE

1. If \( R \) is defined by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), where \( g_1 \) and \( g_2 \) are continuous on \([a, b]\), then the area of \( R \) is given by

\[
A = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} \, dy \, dx.
\]

   Figure 14.2 (vertically simple)

2. If \( R \) is defined by \( c \leq y \leq d \) and \( h_1(y) \leq x \leq h_2(y) \), where \( h_1 \) and \( h_2 \) are continuous on \([c, d]\), then the area of \( R \) is given by

\[
A = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} \, dx \, dy.
\]

   Figure 14.3 (horizontally simple)

### NOTE

Be sure you see that the orders of integration of these two integrals are different—the order \( dy \, dx \) corresponds to a vertically simple region, and the order \( dx \, dy \) corresponds to a horizontally simple region.
EXAMPLE 3 The Area of a Rectangular Region

Use an iterated integral to represent the area of the rectangle shown in Figure 14.4.

Solution The region shown in Figure 14.4 is both vertically simple and horizontally simple, so you can use either order of integration. By choosing the order $dy \, dx$, you obtain the following:

$$
\int_a^b \int_c^d dy \, dx = \int_a^b \left[ \int_c^d y \right] \, dx
$$

Integrate with respect to $y$.

$$
= \int_a^b (d - c) \, dx
$$

Integrate with respect to $x$.

$$
= \left[ (d - c)x \right]_a^b
$$

$$
= (d - c)(b - a) = \text{(Length)} \cdot \text{(Width)}
$$

Notice that this answer is consistent with what you know from geometry.

Figure 14.4
Try It 4

Use an iterated integral to find the area of the region bounded by the graphs of

\[ f(x) = x \quad \text{Forms upper boundary.} \]
\[ g(x) = x^{3/2} \quad \text{Forms lower boundary.} \]

between \( x = 0 \) and \( x = 1 \).

\[
\text{Area} = \int_0^1 \int_{x^{3/2}}^x dy \, dx
\]

\[
\int_0^1 \int_{x^{3/2}}^x dy \, dx = \int_0^1 \left[ y \right]_{x^{3/2}}^x \, dx
\]

\[
= \int_0^1 \left( x - x^{3/2} \right) \, dx
\]

\[
= \left[ \frac{x^2}{2} - \frac{2x^{5/2}}{5} \right]_0^1
\]

\[
= \left( \frac{1}{2} - \frac{2}{5} \right) - \left( 0 - 0 \right)
\]

\[
= \frac{1}{10}
\]

NOTE: The region of integration of an iterated integral need not have any straight lines as boundaries. For instance, the region of integration shown in Figure 14.5 is vertically simple even though it has no vertical lines as left and right boundaries. The quality that makes the region vertically simple is that it is bounded above and below by the graphs of functions of \( x \).
One order of integration will often produce a simpler integration problem than the other order. For instance, try reworking Example 4 with the order $dx\, dy$—you may be surprised to see that the task is formidable. However, if you succeed, you will see that the answer is the same. In other words, the order of integration affects the ease of integration, but not the value of the integral.

**Try It 5**

Sketch the region whose area is represented by the integral

\[
\int_0^4 \int_{\sqrt{x}}^2 dy\, dx.
\]

Then find another iterated integral using the order $dx\, dy$ to represent the same area and show that both integrals yield the same value.
Sometimes it is not possible to calculate the area of a region with a single iterated integral. In these cases you can divide the region into subregions such that the area of each subregion can be calculated by an iterated integral. The total area is then the sum of the iterated integrals.

**Try It 6**

Find the area of the region $R$ that lies below

$$y = \frac{16}{x}$$

above the $x$-axis, to the left of the line $x = 8$, and below the line

$$y = x$$

as shown in the figure.

\[
\int_0^4 \int_0^{16/x} dy \, dx + \int_4^8 \int_0^{16/x} dy \, dx
\]

\[
\left[ \frac{x^2}{2} \right]_0^4 = 8
\]

\[
\left[ \frac{16}{x} \right]_0^4 = 4
\]

\[
\left[ \frac{16}{x} - 0 \right]_0^4 = 16
\]

\[
\left[ \frac{16}{x} \ln x \right]_0^4 = 16 (\ln 8 - \ln 4)
\]

\[
8 + 16 (\ln 2)
\]

\[
19.09
\]

**NOTE** In Examples 3 to 6, be sure you see the benefit of sketching the region of integration. You should develop the habit of making sketches to help you determine the limits of integration for all iterated integrals in this chapter.
14.1 Exercises

In Exercises 1–10, evaluate the integral.

1. \( \int_1^2 (x + 2y) \, dy \)

2. \( \int_0^1 x \, dy \)

3. \( \int_0^1 x \, dx \), \( y > 0 \)

4. \( \int_0^1 y \, dx \)

5. \( \int_0^1 x^2 \, dx \)

6. \( \int_0^1 (x^2 + 3y^2) \, dy \)

7. \( \int_0^1 \frac{1}{x} \, dx \), \( y > 0 \)

8. \( \int_0^1 (x^2 + y^2) \, dx \)

9. \( \int_0^1 ye^{-y^2} \, dy \)

10. \( \int_0^1 \sin^3 x \cos y \, dx \)

In Exercises 11–30, evaluate the iterated integral.

11. \( \int_0^1 \int_0^x (x + y) \, dy \, dx \)

12. \( \int_0^1 \int_0^y (x^2 - y^2) \, dy \, dx \)

13. \( \int_0^1 \int_0^y (x^2 - 2xy^2) \, dy \, dx \)

14. \( \int_0^1 \int_0^y (x + y^2) \, dy \, dx \)

15. \( \int_0^1 \int_0^y \cos x \, dy \, dx \)

16. \( \int_0^1 \int_0^y e^{x+y} \, dy \, dx \)

17. \( \int_0^1 \int_0^y \cos x \, dy \, dx \)

18. \( \int_0^1 \int_0^y 2xe^{-y} \, dy \, dx \)

19. \( \int_0^1 \int_0^y \sqrt{1 - x^2} \, dy \, dx \)

20. \( \int_0^1 \int_0^y \sqrt{64 - x^2} \, dy \, dx \)

21. \( \int_0^1 \int_0^y (3 + x^2 + \frac{1}{4}y^2) \, dy \, dx \)

22. \( \int_0^1 \int_0^y (10 + 2x^2 + 2y^2) \, dy \, dx \)

23. \( \int_0^1 \int_0^y (x + y) \, dy \, dx \)

24. \( \int_0^1 \int_0^y 3y \, dy \, dx \)

25. \( \int_0^1 \int_0^y \frac{2}{x} \, dy \, dx \)

26. \( \int_0^1 \int_0^y \frac{1}{x^2 + y^2} \, dy \, dx \)

27. \( \int_0^1 \int_0^y \sqrt{a^2 + \theta} \, dy \, dx \)

28. \( \int_0^1 \int_0^y \theta \, dy \, dx \)

29. \( \int_0^1 \int_0^y \theta \, dy \, dx \)

30. \( \int_0^1 \int_0^y 3r^2 \sin \theta \, dy \, dx \)

In Exercises 31–34, evaluate the improper iterated integral.

31. \( \int_0^1 \int_0^\infty \frac{1}{y} \, dy \, dx \)

32. \( \int_0^1 \int_0^\infty \frac{1}{y} \, dy \, dx \)

33. \( \int_1^\infty \int_1^\infty \frac{1}{xy} \, dy \, dx \)

34. \( \int_1^\infty \int_1^\infty \frac{1}{x^y} \, dy \, dx \)

In Exercises 35–38, use an iterated integral to find the area of the region.

35. 

\[
\begin{array}{c|c|c}
\hline
x & y & (x, y) \\
\hline
1 & 2 & (1, 2) \\
2 & 3 & (2, 3) \\
\hline
\end{array}
\]

36. 

\[
\begin{array}{c|c|c}
\hline
x & y & (x, y) \\
\hline
0 & 1 & (0, 1) \\
1 & 1 & (1, 1) \\
2 & 1 & (2, 1) \\
\hline
\end{array}
\]

37. \( y = 4 - x^2 \)

38. \( y = \frac{1}{\sqrt{x-1}} \)

In Exercises 39–46, use an iterated integral to find the area of the region bounded by the graphs of the equations.

39. \( \sqrt{x} + \sqrt{y} = 2 \), \( x = 0 \), \( y = 0 \)

40. \( y = x^{3/2} \), \( y = 2x \)

41. \( 2x - 3y = 0 \), \( x + y = 5 \), \( y = 0 \)

42. \( xy = 9 \), \( y = x \), \( y = 0 \), \( x = 9 \)

43. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

44. \( y = x \), \( y = 2x \), \( x = 2 \)

45. \( y = 4 - x^2 \), \( y = x + 2 \)

46. \( x^2 + y^2 = 4 \), \( x = 0 \), \( y = 0 \)

In Exercises 47–54, sketch the region \( R \) of integration and switch the order of integration.

47. \( \int_0^1 \int_0^x f(x, y) \, dy \, dx \)

48. \( \int_0^1 \int_0^y f(x, y) \, dy \, dx \)

49. \( \int_0^1 \int_0^y \sqrt{x+y} \, dy \, dx \)

50. \( \int_0^1 \int_0^y \sqrt{x+y} \, dy \, dx \)

51. \( \int_0^1 \int_0^y f(x, y) \, dy \, dx \)

52. \( \int_0^1 \int_0^y f(x, y) \, dy \, dx \)

53. \( \int_0^1 \int_0^y f(x, y) \, dy \, dx \)

54. \( \int_0^1 \int_0^y f(x, y) \, dy \, dx \)

In Exercises 55–64, sketch the region \( R \) whose area is given by the iterated integral. Then switch the order of integration and show that both orders yield the same area.

55. \( \int_0^1 \int_0^2 f(x, y) \, dy \, dx \)

56. \( \int_0^1 \int_0^2 f(x, y) \, dy \, dx \)

57. \( \int_0^1 \int_0^2 f(x, y) \, dy \, dx \)

58. \( \int_0^1 \int_0^2 f(x, y) \, dy \, dx \)
65. Think About It Give a geometric argument for the equality.
Verify the equality analytically.
\[
\int_0^1 \int_{\sqrt{50-x^2}}^{5-x} x^2 + y^2 \, dy \, dx = \\
\int_0^1 \int_0^{\sqrt{50-x^2}} x^2 + y^2 \, dy \, dx + \int_{\sqrt{50}}^0 \sqrt{50-y} \, x^2 \, dx \, dy
\]

66. Think About It Complete the iterated integrals so that each one represents the area of the region \( R \) (see figure). Then show that both integrals yield the same area.
(a) Area = \( \int \int_R \, dx \, dy \) 
(b) Area = \( \int \int_R \, dy \, dx \)

In Exercises 67–72, sketch the region of integration. Then evaluate the iterated integral. (Note that it is necessary to switch the order of integration.)

67. \( \int_0^2 \int_0^{\sqrt{1+y^2}} x^2 \, dy \, dx \)
68. \( \int_0^2 \int_0^{\sqrt{2+y^2}} \frac{3}{2+y} \, dy \, dx \)
69. \( \int_0^1 \int_{2e^y}^{3e^y} 4e^y \, dy \, dx \)
70. \( \int_0^1 \int_0^{e^{-y}} e^{-y} \, dy \, dx \)
71. \( \int_0^\pi \int_0^\pi \sin \theta \, \sin \theta \, dx \, dy \)
72. \( \int_0^1 \int_0^1 \sqrt{x} \sin x \, dx \, dy \)

In Exercises 73–76, use a computer algebra system to evaluate the iterated integral.
73. \( \int_0^1 \int_0^{2\pi} (x^2 + 3\pi^2) \, dx \, dy \)
74. \( \int_0^1 \int_0^{2\pi} \sin(x + y) \, dx \, dy \)
75. \( \int_0^1 \int_0^{2\pi} \frac{2}{(x + 1)(y + 1)} \, dx \, dy \)
76. \( \int_0^1 \int_0^{2\pi} (x^2 + y^2) \, dx \, dy \)

In Exercises 77 and 78, (a) sketch the region of integration, (b) switch the order of integration, and (c) use a computer algebra system to show that both orders yield the same value.
77. \( \int_0^1 \int_0^{\sqrt{4y}} (x^2 - xy^3) \, dx \, dy \)
78. \( \int_0^1 \int_0^{\sqrt{16 - x^2}} \frac{2x}{x^2 + y^2 + 1} \, dx \, dy \)

In Exercises 79–82, use a computer algebra system to approximate the iterated integral.
79. \( \int_0^1 \int_0^{e^2} e^y \, dx \, dy \)
80. \( \int_0^1 \int_0^{\sqrt{16 - x^2}} \frac{2x}{x^2 + y^2} \, dx \, dy \)
81. \( \int_0^1 \int_0^{\cos \theta} 2 \, d\theta \, d\theta \)
82. \( \int_0^1 \int_0^{15 \theta} \sin \theta \, d\theta \, d\theta \)

Writing About Concepts
83. Explain what is meant by an iterated integral. How is it evaluated?
84. Describe regions that are vertically simple and regions that are horizontally simple.
85. Give a geometric description of the region of integration if the inside and outside limits of integration are constants.
86. Explain why it is sometimes an advantage to change the order of integration.

True or False? In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.
87. \( \int_0^1 \int_0^{\pi/2} f(x, y) \, dx \, dy = \int_0^{\pi/2} \int_0^1 f(x, y) \, dx \, dy \)
88. \( \int_0^1 \int_0^{\pi/2} f(x, y) \, dx \, dy = \int_0^{\pi/2} \int_0^1 f(x, y) \, dx \, dy \)
\[ y = \frac{1}{\sqrt{x-1}} \quad \text{for} \quad 2 \leq x \leq 5 \]

- \( x = 2 \quad y = \frac{1}{\sqrt{1}} = 1 \)
- \( x = 5 \quad y = 0 \)

\[ \int_0^1 \frac{1}{\sqrt{x-1}} \, dx \]

\[ \int_2^5 (x-1)^{-\frac{1}{2}} \, dx \quad \text{(substitute} \quad u = x-1, \quad du = dx) \]

\[ 2 (x-1)^{\frac{1}{2}} \bigg|_2^5 \]

\[ 2 (4)^{\frac{1}{2}} - 2(1) \]

\[ y - 2 = (2) \]
66. Think About It  Complete the iterated integrals so that each one represents the area of the region $R$ (see figure). Then show that both integrals yield the same area.

(a) Area = $\int \int dx \, dy$  \hspace{1cm} (b) Area = $\int \int dy \, dx$

\[ y = \sqrt{x} \hspace{1cm} y = \frac{x}{2} \hspace{1cm} (4, 2) \]

In Exercises 47–54, sketch the region $R$ of integration and switch the order of integration.