63. \[ \int_{y}^{r} \int_{x}^{y} dx \, dy \]

\[ \int_{0}^{\frac{1}{2}} \int_{x^2}^{\sqrt{x}} \, dy \, dx \]

\[ \int_{0}^{\frac{1}{2}} \int_{x^2}^{\sqrt{x}} \, dx \, dy \]

\[ \int_{0}^{\frac{1}{3}} \int_{x^3}^{\sqrt[3]{x}} \, dy \, dx \]

Consider the region bounded by the curves:

- **Right:** \[ x = y^2 \]  \( \text{at } (1, 1) \)
- **Left:** \[ x = \sqrt[3]{y} \]  \( \text{at } (2, 2) \)

Evaluate the double integrals to find the area:

\[ \int_{0}^{\frac{1}{3}} \left( \sqrt[3]{y} - y^2 \right) dy \]

\[ = \left[ \frac{3}{4} y^\frac{4}{3} - \frac{1}{3} y^3 \right]_{0}^{\frac{1}{3}} \]

\[ = \frac{3}{4} \left( \frac{1}{3} \right)^\frac{4}{3} - \frac{1}{3} \left( \frac{1}{3} \right)^3 \]

\[ = \frac{9}{12} - \frac{1}{12} = \frac{5}{12} \]

Therefore, the area is \( \frac{5}{12} \).
The integral 
\[ \int_{0}^{2} \int_{x}^{2} x \sqrt{1 + y^3} \, dy \, dx \]
represents the volume of the region above the xy plane within the triangle bounded by the lines \( y = x \) and \( y = 2 \), with \( x \geq 0 \).}

The integral can be transformed by setting \( u = 1 + y^3 \), with limits of integration from \( y = 0 \) to \( y = 2 \), resulting in:

\[ \frac{1}{3} \int_{1}^{9} \frac{1}{u^{\frac{3}{2}}} \, du = \frac{52}{3} \cdot \frac{1}{6} = \frac{26}{9} \]

The volume of the region above the xy plane within the triangle is \( \frac{26}{9} \).
Double Integrals and Volume of a Solid Region

You already know that a definite integral over an interval uses a limit process to assign measures to quantities such as area, volume, arc length, and mass. In this section, you will use a similar process to define the double integral of a function of two variables over a region in the plane.

Consider a continuous function \( f \) such that \( f(x, y) \geq 0 \) for all \((x, y)\) in a region \( R \) in the \( xy \)-plane. The goal is to find the volume of the solid region lying between the surface given by

\[ z = f(x, y) \]

and the \( xy \)-plane, as shown in Figure 14.8. You can begin by superimposing a rectangular grid over the region, as shown in Figure 14.9. The rectangles lying entirely within \( R \) form an inner partition \( \Delta \), whose norm \( \| \Delta \| \) is defined as the length of the longest diagonal of the \( n \) rectangles. Next, choose a point \((x_i, y_i)\) in each rectangle and form the rectangular prism whose height is \( f(x_i, y_i) \), as shown in Figure 14.10. Because the area of the \( i \)th rectangle is

\[ \Delta A_i \]

it follows that the volume of the \( i \)th prism is

\[ f(x_i, y_i) \Delta A_i \]

and you can approximate the volume of the solid region by the Riemann sum of the volumes of all \( n \) prisms,

\[ \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i \]

as shown in Figure 14.11. This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles, as shown in Example 1.

---

Figure 14.8

Surface: \( z = f(x, y) \)

Figure 14.9

The rectangles lying within \( R \) form an inner partition of \( R \).

Figure 14.10

Rectangular prism whose base has an area of \( \Delta A_i \) and whose height is \( f(x_i, y_i) \).

Figure 14.11

Volume approximated by rectangular prisms
Try It 1

Approximate the volume of the solid lying between $f(x, y) = xy$ and the rectangular region $R$ given by $0 \leq x < 4$, $0 \leq y < 2$. Use a partition made up of squares whose edges have a length of 1.

$$z = f(x, y) = xy$$

$z$ values

- $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$
- $(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})$
- $(\frac{3}{2}, \frac{1}{2}, \frac{1}{4})$
- $(\frac{3}{2}, \frac{1}{2}, \frac{3}{4})$
- $(\frac{3}{2}, \frac{1}{2}, \frac{5}{4})$
- $(\frac{3}{2}, \frac{1}{2}, \frac{3}{2})$

$$\sum_{i=1}^{6} f(x_i, y_i) \cdot \Delta A_i = \sum_{i=1}^{6} x_i \cdot y_i \cdot \Delta A_i = \frac{41}{9} + \frac{15}{4} + \frac{1}{4} + \frac{1}{4} = \frac{64}{9} \approx 7.1$$

Estimate

$$\int_{0}^{4} \int_{0}^{2} xy \, dy \, dx$$

$$\frac{xy^2}{2} \bigg|_{0}^{2} = \int_{0}^{4} 2x \, dx = x^2 \bigg|_{0}^{4} = (16) - (0) = 16$$
In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

\[
\text{Volume} = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i.
\]

The precise meaning of this limit is that the limit is equal to \( L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\left| L - \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i \right| < \epsilon
\]

for all partitions \( \Delta \) of the plane region \( R \) (that satisfy \( ||\Delta|| < \delta \)) and for all possible choices of \( x_i \) and \( y_i \) in the \( i \)th region.

Using the limit of a Riemann sum to define volume is a special case of using the limit to define a double integral. The general case, however, does not require that the function be positive or continuous.

### Definition of Double Integral

If \( f \) is defined on a closed, bounded region \( R \) in the \( xy \)-plane, then the double integral of \( f \) over \( R \) is given by

\[
\int_R \int f(x, y) \, dA = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i
\]

provided the limit exists. If the limit exists, then \( f \) is integrable over \( R \).

\[\text{Note}\] Having defined a double integral, you will see that a definite integral is occasionally referred to as a single integral.

Sufficient conditions for the double integral of \( f \) on the region \( R \) to exist are that \( R \) can be written as a union of a finite number of nonoverlapping subregions (see Figure 14.14) that are vertically or horizontally simple and that \( f \) is continuous on the region \( R \).

A double integral can be used to find the volume of a solid region that lies between the \( xy \)-plane and the surface given by \( z = f(x, y) \).

### Volume of a Solid Region

If \( f \) is integrable over a plane region \( R \) and \( f(x, y) \geq 0 \) for all \((x, y)\) in \( R \), then the volume of the solid region that lies above \( R \) and below the graph of \( f \) is defined as

\[
V = \int_R \int f(x, y) \, dA.
\]
Properties of Double Integrals

Double integrals share many properties of single integrals.

**THEOREM 14.1 PROPERTIES OF DOUBLE INTEGRALS**

Let \( f \) and \( g \) be continuous over a closed, bounded plane region \( R \), and let \( c \) be a constant.

1. \[ \int_R \left( cf(x, y) \right) dA = c \int_R f(x, y) dA \]

2. \[ \int_R \left[ f(x, y) \pm g(x, y) \right] dA = \int_R f(x, y) dA \pm \int_R g(x, y) dA \]

3. \[ \int_R f(x, y) dA \geq 0, \quad \text{if } f(x, y) \geq 0 \]

4. \[ \int_R f(x, y) dA \geq \int_R g(x, y) dA, \quad \text{if } f(x, y) \geq g(x, y) \]

5. \[ \int_R f(x, y) dA = \int_{R_1} f(x, y) dA + \int_{R_2} f(x, y) dA \], where \( R \) is the union of two nonoverlapping subregions \( R_1 \) and \( R_2 \).

Two regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to \( R_1 \) and \( R_2 \) is 0.

**Figure 14.14**
Evaluation of Double Integrals

Normally, the first step in evaluating a double integral is to rewrite it as an iterated integral. To show how this is done, a geometric model of a double integral is used as the volume of a solid.

Consider the solid region bounded by the plane \( z = f(x, y) = 2 - x - 2y \) and the three coordinate planes, as shown in Figure 14.15. Each vertical cross section taken parallel to the \( yz \)-plane is a triangular region whose base has a length of \( y = (2 - x)/2 \) and whose height is \( z = 2 - x \). This implies that for a fixed \( y \), the area of the triangular cross section is

\[
A(y) = \frac{1}{2} \text{(base)(height)} = \frac{1}{2} \left( \frac{2-x}{2} \right) \left( 2-x \right) = \frac{(2-x)^2}{4}.
\]

By the formula for the volume of a solid with known cross sections (Section 7 volume of the solid is

\[
\text{Volume} = \int_0^b A(x) \, dx
\]

\[
= \int_0^2 \frac{(2-x)^2}{4} \, dx
\]

\[
= \left. \frac{(2-x)^3}{12} \right|_0^2 = \frac{2}{3}.
\]

This procedure works no matter how \( A(x) \) is obtained. In particular, you can find by integration, as shown in Figure 14.16. That is, you consider \( x \) to be constant integrate \( z = 2 - x - 2y \) from 0 to \((2-x)/2\) to obtain

\[
A(x) = \int_0^{(2-x)/2} (2-x-2y) \, dy
\]

\[
= \left[ (2-x)y - y^2 \right]_0^{(2-x)/2} = \frac{(2-x)^2}{4}.
\]

Combining these results, you have the iterated integral

\[
\text{Volume} = \int_0^2 \int_0^{(2-x)/2} (2-x-2y) \, dy \, dx.
\]

To understand this procedure better, it helps to imagine the integrating motions. For the inner integration, a vertical line sweeps out the area of a cross section. For the outer integration, the triangular cross section sweeps out the volume, as shown in Figure 14.17.
The following theorem was proved by the Italian mathematician Guido Fubini (1879–1943). The theorem states that if $R$ is a vertically or horizontally simple region and $f$ is continuous on $R$, the double integral of $f$ on $R$ is equal to an iterated integral.

**THEOREM 14.2 FUBINI'S THEOREM**

Let $f$ be continuous on a plane region $R$.

1. If $R$ is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where $g_1$ and $g_2$ are continuous on $[a, b]$, then

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b f(x, y) \, dx \, dy.$$

2. If $R$ is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where $h_1$ and $h_2$ are continuous on $[c, d]$, then

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_c^d f(x, y) \, dy \, dx.$$
Try It 2

Evaluate
\[ \int \int_R x^2 y^2 \, dA \]
where \( R \) is the region given by \( y \leq x \leq \sqrt{y} \), \( 0 \leq y \leq 1 \).

\[ \int_0^1 \int_0^y x^2 y^2 \, dx \, dy \]

\[ \left[ \frac{x^3 y^2}{3} \right]_0^1 = \frac{1}{3} \int_0^1 y^2 \, dy \]

\[ \left[ \frac{y^3}{3} \right]_0^1 = \frac{1}{3} \int_0^1 y^3 \, dy \]

\[ \left[ \frac{y^4}{4} \right]_0^1 = \frac{1}{4} \int_0^1 y^4 \, dy \]

\[ \left[ \frac{y^6}{6} \right]_0^1 = \frac{1}{6} \int_0^1 y^6 \, dy \]

\[ \frac{2}{9} \left( \frac{1}{3} \cdot \frac{9}{2} \right) - \frac{1}{18} = \frac{1}{54} \]
Try It 3

Find the volume of the solid region bounded by \( z = 1 - x - y \), \( y = 0 \), \( x = 0 \) and the xy-plane, as shown in Figure 1.

Vertical Rectangles

\[ \int_0^1 \int_0^{1-x} (1-x-y) \, dy \, dx \]

\[ u = 1-x \]
\[ du = -dx \]

\[ (-\frac{1}{6}) \left[ \frac{(1-x)^3}{3} \right]_0^1 \]

\[ \text{Volume} \]
In Examples 2 and 3, the problems could be solved with either order of integration because the regions were both vertically and horizontally simple. Moreover, had you used the order $dx \, dy$, you would have obtained integrals of comparable difficulty. There are, however, some occasions in which one order of integration is much more convenient than the other. Example 4 shows such a case.

**Try It 4**

Find the volume of the solid region $R$ bounded by the surface

$$f(x, y) = \sqrt{y} \cos y$$

and $y = x^2$, $y = 1$, $x = 0$, and $x = 2$, as shown in the figure.

$$\int_0^1 2 \sqrt{y} \cos y \, dy = x \sqrt{y} \cos y \Bigg|_0^1 = 2 \sqrt{y} \cos y \Bigg|_0^1 = y \cos y$$

$$\int_0^2 \left[ \int_0^1 (y \sin y + \cos y) \, dy \right] \, dx$$

$$= \int_0^2 \left[ y \sin y + \cos y \right] \Bigg|_0^1 \, dx$$

$$= \int_0^2 (\sin y + \cos y) \, dx$$

$$= \int_0^2 \sin y \, dx + \int_0^2 \cos y \, dx$$

$$= \left[ -\cos y \right]_0^2 + \left[ \sin y \right]_0^1$$

$$= -\cos 2 + \sin 1 - (-\cos 0 + \sin 0)$$

$$= -\cos 2 + \sin 1 + 1$$

$$= -1.077 + 0.841 = -0.236$$

$$\text{Volume} = 5.805$$
Try It 5

Find the volume of the solid region $R$ bounded by $y = 1 - x^2$ and $z = 1 - x^2$, as shown in Figure 1.

\[
\int_0^1 \int_0^{1-x^2} (1-x^2) \, dy \, dx
\]
**Average Value of a Function**

Recall from Section 4.4 that for a function $f$ in one variable, the average value of $f$ on $[a, b]$ is

\[
\frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
\]

Given a function $f$ in two variables, you can find the average value of $f$ over the region $R$ as shown in the following definition.

**DEFINITION OF THE AVERAGE VALUE OF A FUNCTION OVER A REGION**

If $f$ is integrable over the plane region $R$, then the **average value of $f$ over $R$** is

\[
\frac{1}{A} \int_{R} f(x, y) \, dA
\]

where $A$ is the area of $R$.

*Average Value* In Exercises 59–64, find the average value of $f(x, y)$ over the region $R$.

64. $f(x, y) = \sin(x + y)$

   $R$: rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, \pi)$, $(0, \pi)$
14.2 Exercises

**Approximation** In Exercises 1–4, approximate the integral \( \int_D f(x, y) \, dA \) by dividing the rectangle \( D \) with vertices \((0, 0), (4, 0), (4, 2), \) and \((0, 2)\) into eight equal squares and finding the sum \( \sum_{i=1}^{8} f(x_i, y_i) \Delta A \), where \((x_i, y_i)\) is the center of the \(i\)th square.

Evaluate the iterated integral and compare it with the approximation.

1. \( \int_0^2 \int_1^2 (x + y) \, dy \, dx \)
2. \( \frac{1}{2} \int_0^2 \int_0^2 x^2 \, dy \, dx \)
3. \( \int_0^2 \int_0^2 (x^2 + y^2) \, dy \, dx \)
4. \( \int_0^2 \int_0^2 \frac{1}{(x + 1)(y + 1)} \, dy \, dx \)

5. **Approximation** The table shows values of a function \( f \) over a square region \( D \). Divide the region into 16 equal squares and select \((x_i, y_i)\) to be the point in the \(i\)th square closest to the origin. Compare this approximation with that obtained by using the point in the \(i\)th square farthest from the origin.

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<td>16</td>
<td>15</td>
<td>12</td>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

6. **Approximation** The figure shows the level curves of a function \( f \) over a square region \( D \). Approximate the integral using four squares, selecting the midpoint of each square as \((x_i, y_i)\).

\[
\int_0^2 \int_0^2 f(x, y) \, dy \, dx
\]

In Exercises 7–12, sketch the region \( D \) and evaluate the iterated integral \( \int_D f(x, y) \, dA \).

7. \( \int_0^2 \int_0^1 (1 + 2x + 2y) \, dy \, dx \)
8. \( \int_0^{\pi/2} \int_0^1 \sin^2 x \cos^2 y \, dy \, dx \)
9. \( \int_0^1 \int_0^{1/y} (x + y) \, dx \, dy \)
10. \( \int_0^1 \int_0^{1/y^2} x^2 \, dx \, dy \)
11. \( \int_0^1 \int_0^{\sqrt{1-y^2}} (x + y) \, dx \, dy \)
12. \( \int_0^2 \int_0^2 e^{x+y} \, dy \, dx + \int_0^1 \int_0^{1-y} e^{x+y} \, dx \, dy \)
In Exercises 13–20, set up integrals for both orders of integration, and use the more convenient order to evaluate the integral over the region $R$.

13. \[ \int_R xy \, dA \]
   $R$: rectangle with vertices $(0, 0)$, $(0, 5)$, $(3, 5)$, $(3, 0)$

14. \[ \int_R \sin x \sin y \, dA \]
   $R$: rectangle with vertices $(-\pi, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(-\pi, \pi/2)$

15. \[ \int_R \frac{y}{x^2 + y^2} \, dA \]
   $R$: trapezoid bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$

16. \[ \int_R x e^2 \, dA \]
   $R$: triangle bounded by $y = 4 - x$, $y = 0$, $x = 0$

17. \[ \int_R -2y \, dA \]
   $R$: region bounded by $y = 4 - x^2$, $y = 4 - x$

18. \[ \int_R \frac{1}{x + 1} \, dA \]
   $R$: region bounded by $y = 0$, $y = \sqrt{x}$, $x = 4$

19. \[ \int_R x \, dA \]
   $R$: sector of a circle in the first quadrant bounded by $y = \sqrt{25 - x^2}$, $3x - 4y = 0$, $y = 0$

20. \[ \int_R (x^2 + y^2) \, dA \]
   $R$: semicircle bounded by $y = \sqrt{4 - x^2}$, $y = 0$

In Exercises 21–30, use a double integral to find the volume of the indicated solid.

21. 

22. 

23. 

24. 

25. \[ 2x + 3y + z = 12 \]

26. \[ x + y + z = 2 \]

27. 

28. 

29. Improper integral

30. Improper integral

In Exercises 21 and 32, use a computer algebra system to find the volume of the solid.

31. 

32. \[ x^2 + z^2 = 1 \]

In Exercises 33–40, set up and evaluate a double integral to find the volume of the solid bounded by the graphs of the equations.

33. \[ z = xy, \ z = 0, \ y = x, \ x = 1, \ first \ quadrant \]

34. \[ y = 0, \ z = 0, \ y = x, \ z = x, \ x = 0, \ x = 5 \]

35. \[ z = 0, \ z = x^2, \ x = 0, \ x = 2, \ y = 0, \ y = 4 \]

36. \[ x^2 + y^2 + z^2 = r^2 \]
37. \( z^2 + z^2 = 1, z^2 + z^2 = 1 \), first octant
38. \( y = 4 - x^2, z = 4 - x^2 \), first octant
39. \( z = x + y, z^2 + z^2 = 4, \) first octant
40. \( z = \frac{2}{x + y}, x = 0, y = 0 \)

In Exercises 41–46, set up a double integral to find the volume of the solid region bounded by the graphs of the equations. Do not evaluate the integral.

41. \( z = x^2 + y^2, z = 4 \)
42. \( z = x^2 + y^2, z = 18 - x^2 - y^2 \)
43. \( z = x^2 + y^2, z = 8x^2 - y^2 \)
44. \( z = x^2 + y^2, z = 8 - y^2 \), first octant
45. \( z = 1 - x^2 - y^2 \)
46. \( z = 1 - x^2 - y^2, z = 6 - y^2 \)

In Exercises 47–50, use a computer algebra system to find the volume of the solid bounded by the graphs of the equations.

47. \( z = 9 - x^2 - y^2 \)
48. \( x^2 + y^2 = 1, z = 0 \), first octant
49. \( z = 1 + x + y, z = 0, x = 0, y = 0, z = 0, x = 0, z = 0, x = 0, z = 0 \)

72. The following iterated integrals represent the solution to the same problem. Which iterated integral is easier to evaluate? Explain your reasoning.

(a) \( \int_0^1 \int_0^1 x^2 y^2 dx \) (b) \( \int_0^1 \int_0^1 y^2 x^2 dx \)

In Exercises 73–76, show that the function is a joint density function and find the required probability.

73. \( f(x, y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)
74. \( f(x, y) = \begin{cases} 2xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)
75. \( f(x, y) = \begin{cases} 229 - x^2 y^2 & \text{if } 0 \leq x \leq 3, 0 \leq y \leq 3 \\ 0 & \text{elsewhere} \end{cases} \)
76. \( f(x, y) = \begin{cases} e^{-x^2 - y^2} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)

77. Approximation The base of a pile of sand at a cement plant is rectangular with approximate dimensions of 20 meters by 30 meters. If the base is placed on the xy-plane with one vertex at the origin, the coordinates on the surface of the pile are \( (0, 5, 0), (15, 5, 0), (15, 10, 0), (5, 10, 0), \) and \( (5, 0, 0) \). Approximate the volume of sand in the pile.

78. Programming Consider a continuous function \( f(x, y) \) over the rectangular region \( R \) with vertices \( (a, b), (a, c), (d, b), \) and \( (d, c) \), where \( a < b \) and \( c < d \). Partition the intervals \( [a, b] \) and \( [c, d] \) into \( n \) and \( m \) subintervals, so that the subintervals in a given direction are of equal length. Write a program for a graphics utility to compute the sum

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} f(x, y) dx \]

where \((x, y)\) is the center of a representative rectangle in \( R \).

93. In Exercises 79–82, use a computer algebra system to approximate the iterated integral, and do not use the program in Exercise 78 to approximate the iterated integral for the given values of \( m \) and \( n \).

94. Show that if \( f > 0 \) the function does not have a real-valued integral in the closed interval \( 0 < a < 1, a \in (0, 1) \).

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