14.3 Change of Variables: Polar Coordinates

- Write and evaluate double integrals in polar coordinates.

Double Integrals in Polar Coordinates

Some double integrals are much easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles, cardioids, and rose curves, and for integrands that involve $x^2 + y^2$.

In Section 10.4, you learned that the polar coordinates $(r, \theta)$ of a point are related to the rectangular coordinates $(x, y)$ of the point as follows.

\[
\begin{aligned}
    x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2 \quad \text{and} \quad \tan \theta &= \frac{y}{x}
\end{aligned}
\]

EXAMPLE 1 Using Polar Coordinates to Describe a Region

Use polar coordinates to describe each region shown in Figure 14.24.

(a) \hspace{1cm} (b)

Solution

a. The region $R$ is a quarter circle of radius 2. It can be described in polar coordinates as

$$R = \{(r, \theta); \ 0 \leq r \leq 2, \ 0 \leq \theta \leq \pi/2\}.$$ 

b. The region $R$ consists of all points between concentric circles of radii 1 and 3. It can be described in polar coordinates as

$$R = \{(r, \theta); \ 1 \leq r \leq 3, \ 0 \leq \theta \leq 2\pi\}.$$
Try It 1

Use polar coordinates to describe the region shown in the figure.

\[ r = 5 \sin \theta \]
\[ 0 \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq \theta \leq 5 \sin \theta \]
The regions in Example 1 are special cases of polar sectors

\[ R = \{(r, \theta): r_1 \leq r \leq r_2, \ \theta_1 \leq \theta \leq \theta_2\} \]

as shown in Figure 14.25.
To define a double integral of a continuous function \( z = f(x, y) \) in polar coordinates, consider a region \( R \) bounded by the graphs of \( r = g_1(\theta) \) and \( r = g_2(\theta) \) and the lines \( \theta = \alpha \) and \( \theta = \beta \). Instead of partitioning \( R \) into small rectangles, use partition of small polar sectors. On \( R \), superimpose a polar grid made of rays as circular arcs, as shown in Figure 14.26. The polar sectors \( R_j \) lying entirely within form an inner polar partition \( \Delta \), whose norm \( \|\Delta\| \) is the length of the longe diagonal of the \( n \) polar sectors.

Consider a specific polar sector \( R_p \) as shown in Figure 14.27. It can be shown (as Exercise 75) that the area of \( R \) is

\[
\Delta A = \sum_{j=1}^{n} \Delta r_i \Delta \theta_j
\]

where \( \Delta r_i = r_i - r_{i-1} \) and \( \Delta \theta_j = \theta_j - \theta_{j-1} \). This implies that the volume of the solid height \( f(r_i \cos \theta_j, r_i \sin \theta_j) \Delta r_i \Delta \theta_j \) is approximately

\[
\int_{R} f(r \cos \theta, r \sin \theta) dr \ d\theta = \text{Volume}
\]

The sum on the right can be interpreted as a Riemann sum for \( f(r \cos \theta, r \sin \theta) r \) over the region \( R \). The region \( R \) corresponds to a horizontally simple region \( S \) in the \( r\theta \)-plane, as shown in Figure 14.28. The polar sectors \( R_j \) correspond to rectangles \( S_j \) and the area \( \Delta A_j \) of \( S_j \) is \( \Delta r_i \Delta \theta_j \). So, the right-hand side of the equation corresponds to the double integral

\[
\int_{S} f(r \cos \theta, r \sin \theta) r \ dr \ d\theta.
\]

From this, you can apply Theorem 14.2 to write

\[
\int_{S} f(x, y) \ dA = \int_{R} f(r \cos \theta, r \sin \theta) r \ dr \ d\theta
\]

This suggests the following theorem, the proof of which is discussed in Section 14.8.
THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM

Let \( R \) be a plane region consisting of all points \((x, y) = (r \cos \theta, r \sin \theta)\) satisfying the conditions \(0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta,\) where \(0 \leq (\beta - \alpha) \leq 2\pi.\) If \(g_1\) and \(g_2\) are continuous on \([\alpha, \beta]\) and \(f\) is continuous on \(R,\) then

\[
\int_0^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
\]

**NOTE.** If \(z = f(x, y)\) is nonnegative on \(R,\) then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of \(f\) and the region \(R.\) When using the integral in Theorem 14.3, be certain not to omit the extra factor of \(r\) in the integrand.

The region \(R\) is restricted to two basic types, \(r\)-simple regions and \(\theta\)-simple regions, as shown in Figure 14.29.

\(r\)-Simple region

- Fixed bounds for \(\theta:\)
  - \(\alpha \leq \theta \leq \beta\)
- Variable bounds for \(r:\)
  - \(0 \leq g_1(\theta) \leq r \leq g_2(\theta)\)

\(\theta\)-Simple region

- Variable bounds for \(\theta:\)
  - \(0 \leq h_1(r) \leq \theta \leq h_2(r)\)
- Fixed bounds for \(r:\)
  - \(r_1 \leq r \leq r_2\)
Try It 2

Use polar coordinates to set up and evaluate the double integral \( \iint_R e^{-(x^2+y^2)} \, dA \) that is bounded by the region \( R: x^2 + y^2 \leq 4, x \geq 0, \text{ and } y \geq 0 \).

Set up In Rectangular First

Vertical Rectangles

\[
\iint_R e^{-(x^2+y^2)} \, dy \, dx
\]

Go To Polar

\[
\iint_R e^{-r^2} \, r \, dr \, d\theta
\]

\[
\left[ -\frac{1}{2} \int_0^2 e^{-r^2} \, (2r) \, dr \right]
\]

\[
\frac{1}{2} e^{-r^2} \bigg|_0^2 = -\frac{1}{2} \left[ e^{-4} - e^0 \right]
\]

\[
= -\frac{1}{2} (e^{-4} - 1)
\]

\[
-\frac{1}{2} \int_0^\pi (e^{-r^2} - 1) \, d\theta
\]

\[
-\frac{1}{2} (e^{-r^2} - 1) \bigg|_0^\pi = \left( \frac{\pi}{2} \right) \left( \frac{1}{e^4} - 1 \right)
\]

Volume

\[
= -\frac{\pi}{2} \left( e^{-4} - 1 \right)
\]
In Example 2, be sure to notice the extra factor of \( r \) in the integrand. This comes from the formula for the area of a polar sector. In differential notation, you can write
\[
dA = r \, dr \, d\theta
\]
which indicates that the area of a polar sector increases as you move away from the origin.

**Try It 3**

Use polar coordinates to find the volume of the solid region bounded by
\[
z = x^2 + y^2 + 1 \quad (z = 0) \quad \text{and} \quad x^2 + y^2 = 4.
\]

Rectangular Form
\[
4 \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} (x^2+y^2) \, dy \, dx
\]

All 4 Quadrants
\[
2 \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} (x^2+y^2) \, dy \, dx
\]

Go To Polar
\[
\int_0^{2\pi} \int_0^2 (r^2+1) \, r \, dr \, d\theta
\]
\[
\int_0^{2\pi} \int_0^2 (r^3 + r) \, dr \, d\theta
\]

\[
\left[ \frac{r^4}{4} + \frac{r^2}{2} \right]_0^2 = \left( \frac{16}{4} + \frac{4}{2} \right) - 0 = 6
\]

\[
6 \int_0^{2\pi} \, d\theta = 6\theta \bigg|_0^{2\pi} = 6(2\pi) - 0 = \pi \, \text{Volume}
\]
Try It 4

Use a double integral to find the area enclosed by the graph of \( r = 3 - 2 \sin \theta \).

Solution

Let \( R \) be the curve shown in the figure.

\[
\frac{1}{2} \int_{0}^{2\pi} \int_{0}^{3-2\sin\theta} (r) \, dr \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( (3-2\sin\theta)^2 - 0 \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} (9 - 12\sin\theta + 4\sin^2\theta) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 9 - 12\sin\theta + 4\sin^2\theta \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 9 - 12\sin\theta + 4\sin^2\theta \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 11 - 12\sin\theta - 2\cos^2\theta \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 11 - 12\sin\theta - 2\cos^2\theta \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 11\theta + 2\cos\theta - \sin\theta \right) \, d\theta
\]

\[
\frac{1}{2} \int_{0}^{2\pi} \left( 11\theta + 2\cos\theta - \sin\theta \right) \, d\theta
\]

\[
\frac{1}{2} \left( 2\pi + 2 - 0 \right) - \left( 0 + 12 - 0 \right)
\]

\[
\frac{1}{2} \left( 2\pi \right) = \boxed{11\pi}
\]
\[ \int_0^\pi \int_0^1 r^2 \, dr \, d\theta \]

\[ \sin \theta \]

\[ \int_0^\pi \sin \theta \, d\theta \]

\[ r^2 \, dr \, d\theta \]

\[ \frac{r^3}{3} \sin \theta \]

\[ \frac{r^3}{3} \sin \theta \bigg|_0^\infty \]

\[ \frac{1}{3} \left( -1 \right) \int_0^\pi \left( 1 - \cos^2 \theta \right) \sin \theta \, d\theta \]

\[ \frac{1}{3} \int_0^\pi \left( 1 - u^2 \right) \, du \]

\[ \frac{-1}{3} \left[ u - \frac{u^3}{3} \right]_0^\pi \]

\[ \frac{-1}{3} \left[ \cos \pi \right] - \frac{1}{3} \left[ \cos 0 \right] \]

\[ \frac{-1}{3} \left[ 1 - \left( \frac{1}{3} \right) \right] + \frac{1}{3} \left[ 1 - \frac{1}{3} \right] \]

\[ \frac{-1}{3} \left( -\frac{2}{3} \right) + \left( \frac{1}{3} \times \frac{2}{5} \right) = \frac{8}{9} \]
\[ \frac{1}{4} A = \int_a^b \frac{b}{a^2-x^2} \, dx \]

This substitution -

Let \( u = x = a \sin \theta \)

\[ dx = a \cos \theta \, d\theta \]

\[ \sqrt{a^2-x^2} = a \cos \theta \]

If \( x = 0 \)

\[ u = x = 0 = a \sin \theta \]

\[ \sin^{-1}(0) = 0 \]

If \( x = a \)

\[ u = x = a = a \sin \theta \]

\[ \frac{1}{2} \int \cos^2 \theta \, d\theta \]

\[ \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right] \]

\[ \frac{a b}{2} \left[ \frac{\pi}{2} + \frac{1}{2} \sin \pi \right] - (0 + 0) \]

\[ \frac{ab \pi}{2} + \frac{a b \pi}{2} = \frac{ab \pi}{2} \]

So Area = \( 4 \left( \frac{ab \pi}{2} \right) = ab \pi \)
So far in this section, all of the examples of iterated integrals in polar form have been of the form

$$
\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta
$$

in which the order of integration is with respect to $r$ first. Sometimes you can obtain a simpler integration problem by switching the order of integration, as illustrated in the next example.

**Try It 5**

Find the area of the region bounded above by the spiral

$$r = \frac{\pi}{2\theta}$$

and below by the polar axis, between $r = 1$ and $r = 3$.

The area $A$ can be calculated as

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r \, dr \, d\theta$$

where $r = f(x, y)$.

For the given spiral, we have

$$r = \frac{\pi}{2\theta}$$

and the limits of integration are

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 3$$

The area $A$ is then

$$A = \int_{0}^{\pi/2} \int_{1}^{3} r \, dr \, d\theta$$

Evaluating this integral gives

$$A = \frac{\pi}{2} (3^2 - 1) = \frac{\pi}{2} (8) = 4\pi$$
14.3 Exercises

In Exercises 1–4, the region $R$ for the integral $\iint_R f(x, y) \, dA$ is shown. State whether you would use rectangular or polar coordinates to evaluate the integral.

1. 

```
1 2 3 4
0 0 0 0
```

2. 

```
0 0 0 0
0 0 0 0
```

3. 

```
1 0 0 0
0 0 0 0
```

4. 

```
0 0 0 0
0 0 0 0
```

In Exercises 5–8, use polar coordinates to describe the region shown.

5. 

```
0 0 0 0
0 0 0 0
```

6. 

```
0 0 0 0
0 0 0 0
```

7. 

```
0 0 0 0
0 0 0 0
```

8. 

```
0 0 0 0
0 0 0 0
```

In Exercises 9–16, evaluate the double integral $\iint_R f(r, \theta) \, dr \, d\theta$, and sketch the region $R$.

9. $\int_0^a \int_0^{\pi / 2} r^2 \sin \theta \, d\theta \, dr$

10. $\int_0^a \int_0^{\pi / 2} r^2 \, d\theta \, dr$

11. $\int_0^a \int_0^{\pi / 2} 3r^2 \sin \theta \, d\theta \, dr$

12. $\int_0^a \int_0^{\pi / 2} r^3 \sin \theta \cos \theta \, d\theta \, dr$

13. $\int_0^a \int_0^{\pi / 2} \sqrt{9 - r^2} \, r \, d\theta \, dr$

14. $\int_0^a \int_0^{\pi / 2} re^{-r^2} \, d\theta \, dr$

15. $\int_0^a \int_0^{\pi / 2} \theta \, r \, d\theta \, dr$

16. $\int_0^a \int_0^{\pi / 2} (\sin \theta) r \, d\theta \, dr$

In Exercises 17–26, evaluate the iterated integral by converting to polar coordinates.

17. $\int_0^a \int_0^{\pi / 2} y \, d\theta \, dx$

18. $\int_0^a \int_0^{\pi / 2} x \, d\theta \, dx$

19. $\int_0^a \int_0^{\pi / 2} (x^2 + y^2) \, d\theta \, dx$

20. $\int_0^a \int_0^{\pi / 2} (x^2 + y^2) \, d\theta \, dx$

21. $\int_0^a \int_0^{\pi / 2} (x^2 + y^2)^{3/2} \, d\theta \, dx$

22. $\int_0^a \int_0^{\pi / 2} \sqrt{x^2 + y^2} \, dx \, d\theta$

23. $\int_0^a \int_0^{\pi / 2} xy \, dx \, d\theta$

24. $\int_0^a \int_0^{\pi / 2} x^2 \, dx \, d\theta$

25. $\int_0^a \int_0^{\pi / 2} \cos(x^2 + y^2) \, d\theta \, dx$

26. $\int_0^a \int_0^{\pi / 2} \sin(x^2 + y^2) \, d\theta \, dx$

In Exercises 27 and 28, combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

27. $\int_0^a \int_0^{\pi / 2} \sqrt{x^2 + y^2} \, dx \, d\theta + \int_0^a \int_0^{\pi / 2} \sqrt{x^2 + y^2} \, dx \, d\theta$

28. $\int_0^a \int_0^{\pi / 2} xy \, dx \, d\theta + \int_0^a \int_0^{\pi / 2} xy \, dx \, d\theta$

In Exercises 29–32, use polar coordinates to set up and evaluate the double integral $\iint_R f(x, y) \, dA$.

29. $f(x, y) = x + y, R: x^2 + y^2 \leq 4, x \geq 0, y \geq 0$

30. $f(x, y) = e^{x+y}/2, R: x^2 + y^2 \leq 25, x \geq 0$

31. $f(x, y) = \arctan \frac{x}{y}, R: x^2 + y^2 \leq 1, x^2 + y^2 \leq 4, 0 \leq y \leq x$

32. $f(x, y) = 9 - x^2 - y^2, R: x^2 + y^2 \leq 9, x \geq 0, y \geq 0$

Volume In Exercises 33–38, use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.

33. $z = xy, x^2 + y^2 = 1$, first octant

34. $z = x^2 + y^2 + 3, z = 0, x^2 + y^2 = 1$

35. $z = \sqrt{x^2 + y^2}, z = 0, x^2 + y^2 = 25$

36. $z = \ln(x^2 + y^2), z = 0, x^2 + y^2 \geq 1, x^2 + y^2 \leq 4$

37. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and inside the cylinder $x^2 + y^2 = 4, x \geq 0$

38. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$
39. **Volume** Find a such that the volume inside the hemisphere 
\[ z = \sqrt{16 - x^2 - y^2} \] and outside the cylinder \( x^2 + y^2 = a^2 \) is one-half the volume of the hemisphere.

40. **Volume** Use a double integral in polar coordinates to find the volume of a sphere of radius \( a \).

41. **Volume** Determine the diameter of a hole that is drilled vertically through the center of the solid bounded by the graphs of the equations \( z = 25e^{-\sqrt{x^2 + y^2}}, z = 0 \), and \( x^2 + y^2 = 16 \) if one-tenth of the volume of the solid is removed.

42. **Machine Design** The surfaces of a double-lobed cam are modeled by the inequalities \( \frac{1}{2} \leq r \leq \frac{1}{2}(1 + \cos \theta) \) and
\[
\frac{1 - 9}{4(x^2 + y^2 + 9)} \leq z \leq \frac{9}{4(x^2 + y^2 + 9)}
\]
where all measurements are in inches.

(a) Use a computer algebra system to graph the cam.

(b) Use a computer algebra system to approximate the perimeter of the polar curve
\[ r = \frac{1}{2}(1 + \cos \theta). \]

This is the distance a roller must travel as it runs against the cam through one revolution of the cam.

(c) Use a computer algebra system to find the volume of steel in the cam.

**Area** In Exercises 43–48, use a double integral to find the area of the shaded region.

43. \[ r = 6 \cos \theta \]

44. \[ r = 2 \]

45. \[ r = 1 + \cos \theta \]

46. \[ r = 2 + \sin \theta \]

47. \[ r = 2 \sin 3\theta \]

48. \[ r = 3 \cos 2\theta \]

**Area** In Exercises 49–54, sketch a graph of the region bounded by the graphs of the equations. Then use a double integral to find the area of the region.

49. Inside the circle \( r = 2 \cos \theta \) and outside the circle \( r = 1 \)

50. Inside the cardioid \( r = 2 + \cos \theta \) and outside the circle \( r = 1 \)

51. Inside the circle \( r = 3 \cos \theta \) and outside the cardioid \( r = 1 + \cos \theta \)

52. Inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 3 \cos \theta \)

53. Inside the rose curve \( r = 4 \sin 3\theta \) and outside the circle \( r = 2 \)

54. Inside the circle \( r = 2 \) and outside the cardioid \( r = 2 - 2 \cos \theta \)

**Writing About Concepts**

55. Describe the partition of the region \( R \) of integration in the \( xy \)-plane when polar coordinates are used to evaluate a double integral.

56. Explain how to change from rectangular coordinates to polar coordinates in a double integral.

57. In your own words, describe \( r \)-simple regions and \( \theta \)-simple regions.

58. Each figure shows a region of integration for the double integral \( \iint_D f(x, y) \, dA \). For each region, state whether horizontal representative elements, vertical representative elements, or polar sectors would yield the easiest method for obtaining the limits of integration. Explain your reasoning.

(a) \[ \]

(b) \[ \]

(c) \[ \]

59. Let \( R \) be the region bounded by the circle \( x^2 + y^2 = 9 \).

(a) Set up the integral \( \iint_R f(x, y) \, dA \).

(b) Convert the integral in part (a) to polar coordinates.

(c) Which integral would you choose to evaluate? Why?

**Capstone**

60. **Think About It** Without performing any calculations, identify the double integral that represents the integral of \( f(x) = x^2 + y^2 \) over a circle of radius 4. Explain your reasoning.

(a) \[ \]

(b) \[ \]

(c) \[ \]

(d) \[ \]
61. **Think About It** Consider the program you wrote to approximate double integrals in rectangular coordinates in Exercise 78, in Section 14.2. If the program is used to approximate the double integral
\[
\int_{A} f(r, \theta) \, dA
\]
in polar coordinates, how will you modify \( f \) when it is entered into the program? Because the limits of integration are constants, describe the plane region of integration.

62. **Approximation** Horizontal cross sections of a piece of ice that broke from a glacier are in the shape of a quarter of a circle with a radius of approximately 50 feet. The base is divided into 20 subregions, as shown in the figure. At the center of each subregion, the height of the ice is measured, yielding the following points in cylindrical coordinates:
\[
(5, \frac{\pi}{6}, 7), (15, \frac{\pi}{6}, 8), (25, \frac{\pi}{6}, 10), (35, \frac{\pi}{6}, 12), (45, \frac{\pi}{6}, 9),
\]
\[
(5, \frac{\pi}{3}, 9), (15, \frac{\pi}{3}, 10), (25, \frac{\pi}{3}, 11), (35, \frac{\pi}{3}, 12), (45, \frac{\pi}{3}, 10),
\]
\[
(5, \frac{2\pi}{3}, 9), (15, \frac{2\pi}{3}, 11), (25, \frac{2\pi}{3}, 12), (35, \frac{2\pi}{3}, 12), (45, \frac{2\pi}{3}, 10)
\]
(a) Approximate the volume of the solid.
(b) Ice weighs approximately 57 pounds per cubic foot. Approximate the weight of the solid.
(c) There are 7.48 gallons of water per cubic foot. Approximate the number of gallons of water in the solid.

63. **Approximation** In Exercises 63 and 64, use a computer algebra system to approximate the iterated integral.
\[
\int_{0}^{3} \int_{0}^{4} f(x, y) \, dx \, dy
\]
\[
\int_{0}^{6} \int_{0}^{15} f(x, y) \, dx \, dy
\]
Approximation In Exercises 65 and 66, determine which value best approximates the volume of the solid between the xy-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and not by performing any calculations.)
65. \( f(x, y) = 15 - 2y; \) the semicircle: \( x^2 + y^2 = 16, \) \( y \geq 0 \)
(a) 100 (b) 200 (c) 300 (d) 400 (e) 800
66. \( f(x, y) = xy + 2; \) the quarter circle: \( x^2 + y^2 = 9, \) \( x \geq 0, y \geq 0 \)
(a) 25 (b) 8 (c) 100 (d) 50 (e) -30

67. **True or False?** In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.
67. If \( \int_{A} f(r, \theta) \, dA > 0, \) then \( f(r, \theta) > 0 \) for all \((r, \theta)\) in \( R.\)
68. If \( f(r, \theta) \) is a constant function and the area of the region \( S \) is twice that of the region \( R, \) then \( 2 \int_{S} f(r, \theta) \, dA = \int_{R} f(r, \theta) \, dA.\)

69. **Probability** The value of the integral \( I = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \) is required in the development of the normal probability density function.
(a) Use polar coordinates to evaluate the improper integral.
\[
I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2/2} \, dy \right)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dA
\]
(b) Use the result of part (a) to determine \( I.\)

70. **FOR FURTHER INFORMATION** For more information on this problem, see the article "Integrating e^x Without Polar Coordinates" by William Durham in *Mathematics Teacher*. To view this article, go to the website www.matharticles.com.

71. **Population** The population density of a city is approximated by the model \( f(x, y) = 4000e^{-100(x^2+y^2)}, x^2 + y^2 \leq 49, \) where \( x \) and \( y \) are measured in miles. Integrate the density function over the indicated circular region to approximate the population of the city.

72. **Probability** Find \( f \) such that the function
\[
f(x, y) = \begin{cases} 
2e^{-x^2+y^2}, & x \geq 0, y \geq 0 \\
0, & \text{elsewhere}
\end{cases}
\]
is a probability density function.

73. **Think About It** Consider the region bounded by the graphs of \( y = 2, y = 4, y = x, \) and \( y = \sqrt{3}x \) and the double integral \( \iint_{R} f \, dA.\) Determine the limits of integration if the region \( R \) is divided into (a) horizontal representative elements, (b) vertical representative elements, and (c) polar sectors.

74. **Repeat Exercise 73** for a region \( R \) bounded by the graph of the equation \( (x - 2)^2 + y^2 = 4.\)

75. **Show that** the area \( A \) of the polar sector \( R \) (see figure) is
\[
A = r \Delta \theta \Delta \theta \Delta r,
\]
where \( r = (r_1 + r_2)/2 \) is the average radius of \( R.\)