\[ Q = \int \int \int (x-xy) \, dx \, dy \, dz \]

\[ 0 \leq x \leq 1 \]

\[ y = 1-x^2 \]

\[ 0 \leq z \leq 6 \]

**Diagram:**

- Upper boundary: \( y = 1 - x^2 \)
- Lower boundary: \( y = 0 \)
- Right boundary: \( x = 1 \)
- Left boundary: \( x = 0 \)
- Front boundary: \( z = 0 \)
- Back boundary: \( z = 6 \)

**Integrals:**

- Vertical: \( \int \int x \, dz \, dy \, dx \)
- Horizontal: \( \int \int x \, dx \, dz \, dy \)
- Front: \( \int \int x \, dx \, dy \, dz \)
- Back: \( \int \int x \, dy \, dz \, dx \)
SECTION 14.6 Triple Integrals and Applications

The following version of Fubini’s Theorem describes a region that is considered simple with respect to the order \( dz \, dy \, dx \). Similar descriptions can be given for the other five orders.

**THEOREM 14.4 Evaluation by Iterated Integrals**

Let \( f \) be continuous on a solid region \( Q \) defined by

\[ a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x, y) \leq z \leq g_2(x, y) \]

where \( h_1, h_2, g_1, \) and \( g_2 \) are continuous functions. Then,

\[
\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx
\]

To evaluate a triple iterated integral in the order \( dz \, dy \, dx \), hold both \( x \) and \( y \) constant for the innermost integration. Then, hold \( y \) constant for the second integration.
In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region $Q$ whose density is given by the density function $\rho$. The center of mass of a solid region $Q$ of mass $m$ is given by $(\bar{x}, \bar{y}, \bar{z})$, where

$$
m = \iiint_Q \rho(x, y, z) \, dV$$

Mass of the solid

$$M_{yz} = \iiint_Q x\rho(x, y, z) \, dV$$

First moment about $yz$-plane

$$M_{xz} = \iiint_Q y\rho(x, y, z) \, dV$$

First moment about $xz$-plane

$$M_{xy} = \iiint_Q z\rho(x, y, z) \, dV$$

First moment about $xy$-plane

and

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$
\[ I_x = \iiint_Q (y^2 + z^2) \rho(x, y, z) \, dV \]  
Moment of inertia about x-axis

\[ I_y = \iiint_Q (x^2 + z^2) \rho(x, y, z) \, dV \]  
Moment of inertia about y-axis

\[ I_z = \iiint_Q (x^2 + y^2) \rho(x, y, z) \, dV \]  
Moment of inertia about z-axis

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

\[ I_x = I_{xz} + I_{xy}, \quad I_y = I_{yz} + I_{xy}, \quad \text{and} \quad I_z = I_{yz} + I_{xz} \]

where \( I_{xy}, I_{xz}, \) and \( I_{yz} \) are as follows.

\[ I_{xy} = \iiint_Q z^2 \rho(x, y, z) \, dV \]

\[ I_{xz} = \iiint_Q y^2 \rho(x, y, z) \, dV \]

\[ I_{yz} = \iiint_Q x^2 \rho(x, y, z) \, dV \]
Multiple Integration

**Definition of Surface Area**

If $f$ and its first partial derivatives are continuous on the closed region $R$ in the $xy$-plane, then the area of the surface $S$ given by $z = f(x, y)$ over $R$ is given by

$$
\text{Surface area} = \int_R \int dS = \int_R \int \sqrt{1 + \left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2} 
$$

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

- **Length on $x$-axis:** $\int_a^b dx$
- **Arc length in $xy$-plane:** $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2} dx$
Section 7.6 discussed several applications of integration involving a lamina of constant density $\rho$. For example, if the lamina corresponding to the region $R$, as shown in Figure 14.33, has a constant density $\rho$, then the mass of the lamina is given by

$$\text{Mass} = \rho A = \rho \int_R \int dA = \int_R \int \rho \, dA.$$  \hspace{1cm} \text{Constant density}

If not otherwise stated, a lamina is assumed to have a constant density. In this section however, you will extend the definition of the term lamina to include thin plates of variable density. Double integrals can be used to find the mass of a lamina of variable density, where the density at $(x, y)$ is given by the density function $\rho$.

**Definition of Mass of a Planar Lamina of Variable Density**

If $\rho$ is a continuous density function on the lamina corresponding to a plane region $R$, then the mass $m$ of the lamina is given by

$$m = \int_R \int \rho(x, y) \, dA.$$  \hspace{1cm} \text{Variable density}

NOTE Density is normally expressed as mass per unit volume. For a planar lamina, however, density is mass per unit surface area.
Similarly, the moment of mass with respect to the y-axis can be approximated by

\[(\text{Mass})_y \approx \int \rho(x, y) \, \Delta A_i \, dy_i.\]

By forming the Riemann sum of all such products and taking the limits as the norm of \(\Delta\) approaches 0, you obtain the following definitions of moments of mass with respect to the x- and y-axes.

**Moments and Center of Mass of a Variable Density Planar Lamina**

Let \(\rho\) be a continuous density function on the planar lamina \(R\). The moments of mass with respect to the x- and y-axes are

\[M_x = \int \int_R y\rho(x, y) \, dA \quad \text{and} \quad M_y = \int \int_R x\rho(x, y) \, dA.\]

If \(m\) is the mass of the lamina, then the center of mass is

\[(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).\]

If \(R\) represents a simple plane region rather than a lamina, the point \((x, y)\) is called the centroid of the region.

For some planar laminas with a constant density \(\alpha\), you can determine the center...
Moments of Inertia

The moments of $M_x$ and $M_y$ used in determining the center of mass of a lamina are sometimes called the first moments about the $x$- and $y$-axes. In each case, the moment is the product of a mass times a distance.

$$M_x = \int_R (y)\rho(x, y) \, dA$$
$$M_y = \int_R (x)\rho(x, y) \, dA$$

You will now look at another type of moment—the second moment, or the moment of inertia of a lamina about a line. In the same way that mass is a measure of the tendency of matter to resist a change in straight-line motion, the moment of inertia about a line is a measure of the tendency of matter to resist a change in rotational motion. For example, if a particle of mass $m$ is a distance $d$ from a fixed line, its moment of inertia about the line is defined as

$$I = md^2 = \text{(mass)}(\text{distance})^2$$

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the $x$- and $y$-axes of a lamina of variable density. These second moments are denoted by $I_x$ and $I_y$, and in each case the moment is the product of a mass times
\[ I = md^2 = (\text{mass})(\text{distance})^2. \]

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the \( x \)- and \( y \)-axes of a lamina of variable density. These second moments are denoted by \( I_x \) and \( I_y \), and in each case the moment is the product of a mass times the square of a distance.

\[
I_x = \int_R \int (y^2)\rho(x, y) \, dA \\
I_y = \int_R \int (x^2)\rho(x, y) \, dA
\]

Square of distance to \( x \)-axis \hspace{1cm} Mass \hspace{1cm} Square of distance to \( y \)-axis \hspace{1cm} Mass

The sum of the moments \( I_x \) and \( I_y \) is called the \textbf{polar moment of inertia} and is denoted by \( I_p \).

\textbf{EXAMPLE 4} \hspace{1cm} \textbf{Finding the Moment of Inertia}

Find the moment of inertia about the \( x \)-axis of the lamina in Example 3.

\textbf{Solution} \hspace{1cm} From the definition of moment of inertia, you have

\[ r^2 \int \int (4-x^2) \, dA \]
On the other hand, the kinetic energy $E$ of a mass $m$ moving in a straight line at a velocity $v$ is

$$E = \frac{1}{2}mv^2.$$  

Kinetic energy for linear motion

So, the kinetic energy of a mass moving in a straight line is proportional to its mass, but the kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

The radius of gyration $\bar{r}$ of a revolving mass $m$ with moment of inertia $I$ is defined to be

$$\bar{r} = \sqrt{\frac{I}{m}}.$$  

Radius of gyration

If the entire mass were located at a distance $\bar{r}$ from its axis of revolution, it would have the same moment of inertia and, consequently, the same kinetic energy. For instance, the radius of gyration of the lamina in Example 4 about the x-axis is given by

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32.768k/315}{256k/15}} = \sqrt{\frac{128}{21}} \approx 2.469.$$
SECTION 14.3  Change of Variables: Polar Coordinates

THEOREM 14.3  Change of Variables to Polar Form

Let $R$ be a plane region consisting of all points $(x, y) = (r \cos \theta, r \sin \theta)$ satisfying the conditions $0 \leq g_1(\theta) \leq r \leq g_2(\theta)$, $\alpha \leq \theta \leq \beta$, where $0 \leq (\beta - \alpha) \leq 2\pi$. If $g_1$ and $g_2$ are continuous on $[\alpha, \beta]$ and $f$ is continuous on $R$, then

$$\int_R \int f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

NOTE  If $z = f(x, y)$ is nonnegative on $R$, then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of $f$ and the region $R$.

The region $R$ is restricted to two basic types, $r$-simple regions and $\theta$-simple regions, as shown in Figure 14.28.

\[
\begin{align*}
\text{Fixed bounds for } \theta: & \quad \theta = \beta \\
\alpha \leq \theta \leq \beta & \quad \text{Variable bounds for } r \\
0 \leq h_1(r) & \leq \theta \leq h_2(r)
\end{align*}
\]
Multiple Integration

Definition of Double Integral

If $f$ is defined on a closed, bounded region $R$ in the $xy$-plane, then the double integral of $f$ over $R$ is given by

$$\int_R \int f(x, y) \, dA = \lim_{|\Delta| \to 0} \sum_{i=1}^{n} f(x^*_i, y^*_i) \Delta A_i$$

provided the limit exists. If the limit exists, then $f$ is integrable over $R$.

*NOTE* Having defined a double integral, you will see that a definite integral is occasionally referred to as a single integral.

Sufficient conditions for the double integral of $f$ on the region $R$ to exist are that $R$ can be written as a union of a finite number of nonoverlapping subregions (see page 14.14) that are vertically or horizontally simple and that $f$ is continuous on the region $R$.

A double integral can be used to find the volume of a solid region that lies...
Sufficient conditions for the double integral of $f$ on the region $R$ to exist are that $R$ can be written as a union of a finite number of nonoverlapping subregions (see Figure 14.14) that are vertically or horizontally simple and that $f$ is continuous on the region $R$.

A double integral can be used to find the volume of a solid region that lies between the $xy$-plane and the surface given by $z = f(x, y)$.

### Volume of a Solid Region

If $f$ is integrable over a plane region $R$ and $f(x, y) \geq 0$ for all $(x, y)$ in $R$, then the volume of the solid region that lies above $R$ and below the graph of $f$ is defined as

$$V = \iint_R f(x, y) \, dA$$

**Properties of Double Integrals**

Double integrals share many properties of single integrals.
Properties of Double Integrals

Double integrals share many properties of single integrals.

**THEOREM 14.1 Properties of Double Integrals**

Let \( f \) and \( g \) be continuous over a closed, bounded plane region \( R \), and let \( c \) be a constant.

1. \( \iint_R cf(x, y) \, dA = c \iint_R f(x, y) \, dA \)

2. \( \iint_R [f(x, y) \pm g(x, y)] \, dA = \iint_R f(x, y) \, dA \pm \iint_R g(x, y) \, dA \)

3. \( \iint_R f(x, y) \, dA \geq 0 \), if \( f(x, y) \geq 0 \)

4. \( \iint_R f(x, y) \, dA \geq \iint_R g(x, y) \, dA \), if \( f(x, y) \geq g(x, y) \)

5. \( \iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA \), where \( R \) is the union of two nonoverlapping subregions \( R_1 \) and \( R_2 \).
Solution  The base of $R$ in the $xy$-plane is bounded by the lines $y = 0$, $x = 1$, and $y = x$. The two possible orders of integration are shown in Figure 14.21.

$$
\int_0^1 \int_{x}^{1} e^{-y^2} \, dy \, dx
$$

---

Figure 14.21

Setting up the corresponding iterated integrals, you can see that the order $dx \, dy$ requires the antiderivative $e^{-x^2} \, dx$, which is not an elementary function. On the other hand, the order $dy \, dx$ produces the integral

$$
\int_0^1 \int_{x}^{1} e^{-y^2} \, dy \, dx
$$