32) \[ F = -20 \mathbf{k} \]

\[ \mathbf{PQ} = \frac{x}{2} (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}) \]

\[ \text{moment} = \mathbf{PQ} \times F = \begin{vmatrix} i & j & k \\ \frac{x}{2} \cos \phi & \frac{x}{2} \sin \phi & 0 \\ 0 & 0 & -10 \end{vmatrix} \mathbf{i} \mathbf{j} \mathbf{k} \]

\[ \mathbf{PQ} = -10 \cos \phi \mathbf{i} + 0 \mathbf{j} + 10 \mathbf{k} \]

\[ 11 \mathbf{PQ} \times F \mathbf{l} = \sqrt{(-10 \cos \phi)^2 + 0^2 + 10^2} = 10 \text{ lbs.} \]
Section 11.5

(45) Find plane passing through $(1,2,3)$ parallel to $xy$ plane. $z = 0$

$\mathbf{N} = \mathbf{y}$

$\mathbf{N} = \mathbf{k}$

Equation of plane

$D(x-1) + 0(y-2) + 1(z-3) = 0$

$z = 3$

Parallel to $z = 0$ $xy$ plane

Pass through point $(1,2,3)$
Section 11.4

\[ u \cdot (v \times w) \]

\[ u = 1i + j + 0k \]
\[ v = 0i + 0 + 1k \]
\[ w = 1i + 0j + 1k \]

\[ \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \]

\[ \text{Triple Scalar Product} \]

\[ \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \]

\[ 1(1-0) -1(0-1) + 0(0-1) \]

\[ 1 + (-1) \]

\[ 2 \]
a scalar multiple of \( \mathbf{v} \), and you can write \( \overrightarrow{PQ} = \mathbf{v} + t \mathbf{v} \), where \( t \) is a scalar (a real number).

\[
\overrightarrow{PQ} = (x - x_1, y - y_1, z - z_1) = (at, bt, ct) = t \mathbf{v}
\]

By equating corresponding components, you can obtain parametric equations of a line in space.

**THEOREM 11.11  Parametric Equations of a Line in Space**

A line \( L \) parallel to the vector \( \mathbf{v} = (a, b, c) \) and passing through the point \( P(x_1, y_1, z_1) \) is represented by the parametric equations

\[
x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.
\]

If the direction numbers \( a, b, \) and \( c \) are all nonzero, you can eliminate the parameter \( t \) to obtain symmetric equations of the line.

\[
\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}
\]

**EXAMPLE 1  Finding Parametric and Symmetric Equations**
\[ \mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle = \langle a, b \rangle \]

Using the direction numbers \( a = 3, b = 2, \) and \( c = 5 \) with the can obtain the parametric equations

\[ x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t. \]

**Try It**

**Exploration A**

**Open Exploration**

**Planes in Space**

You have seen how an equation of a line in space can be obtained line and a vector *parallel* to it. You will now see that an equation can be obtained from a point in the plane and a vector *normal* plane.

Consider the plane containing the point \( P(x_1, y_1, z_1) \) have vector \( \mathbf{n} = \langle a, b, c \rangle \), as shown in Figure 11.45. This plane \( Q(x, y, z) \) for which vector \( \overrightarrow{PQ} \) is orthogonal to \( \mathbf{n} \). Using the dot the following.
The equation of the plane is given by $3x + 2y + 4z = 12$. You can find the $xy$-trace by letting $z = 0$ and sketching the line $3x + 2y = 12$ in the $xy$-plane. This line intersects the $x$-axis at $(4, 0, 0)$ and the $y$-axis at $(0, 6, 0)$. In Figure 11.49, this process is continued by finding the $yz$-trace and the $xz$-trace, and then shading the triangular region lying in the first octant.

Traces of the plane $3x + 2y + 4z = 12$:
- $xy$-trace ($z = 0$): $3x + 2y = 12$
- $yz$-trace ($x = 0$): $2y + 4z = 12$
- $xz$-trace ($y = 0$): $3x + 4z = 12$
\[2x + z = 1\]

Parallel to \( y \)-Axis

\( x \) Trace

\[2x + z = 1\]

\( x = 0 \) \( (0, 0, y) \)

\( z = 0 \) \( (\frac{1}{2}, 0, 0) \)

Move to right \(-\frac{1}{2}\) and draw a parallel line
If an equation of a plane has a missing variable, the plane must be parallel to the axis represented by the missing variable. Figure 11.50. If two variables are missing from an equation, the plane is parallel to the coordinate plane represented by the missing variables.

Plane $2x + z = 1$ is parallel to the $y$-axis. Figure 11.50.

Plane $ax + d = 0$ is parallel to the $yz$-plane.

Plane $by + d = 0$ is parallel to the $xz$-plane.
$x = \_\_$
Traces of the plane $3x + 2y + 4z = 12$

Figure 11.49

If an equation of a plane has a missing variable, such as $2x + z = 1$, must be parallel to the axis represented by the missing variable, as shown in 11.50. If two variables are missing from an equation of a plane, it is parallel to the coordinate plane represented by the missing variables, as shown in Figure 11.51.

Plane $ax + d = 0$ is parallel to the $yz$-plane.
Plane $by + d = 0$ is parallel to the $xz$-plane.
Plane $cz + d = 0$ is parallel to the $xy$-plane.
Traces of the plane $3x + 2y + 4z = 12$

Figure 11.49

If an equation of a plane has a missing variable, such as $2x + z = 1$, must be parallel to the axis represented by the missing variable, as shown 11.50. If two variables are missing from an equation of a plane, it is parallel coordinate plane represented by the missing variables, as shown in Figure 11.51.
Distances Between Points, Planes, and Lines

This section is concluded with the following discussion on problems involving distance in space.

1. Finding the distance between a point and a plane
2. Finding the distance between a point and a line

The solutions of these problems illustrate the versatility and utility of coordinate geometry: the first problem uses the dot product and the second problem uses the cross product.

The distance \( D \) between a point \( Q \) and a plane is the length of the segment connecting \( Q \) to the plane, as shown in Figure 11.52. To find this distance, you can find this distance by projecting the vector \( \vec{PQ} \) onto \( \vec{n} \). The length of this projection is the desired distance.

**THEOREM 11.13 Distance Between a Point and a Plane**

The distance between a plane and a point \( Q \) (not in the plane)

\[
D = \|\text{proj}_n \vec{PQ}\| = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}
\]
coordinate geometry: the first problem uses the dot product of two vectors, and the second problem uses the cross product.

The distance $D$ between a point $Q$ and a plane is the length of the shortest line segment connecting $Q$ to the plane, as shown in Figure 11.52. If $P$ is any point in the plane, you can find this distance by projecting the vector $\vec{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.

**THEOREM 11.13  Distance Between a Point and a Plane**

The distance between a plane and a point $Q$ (not in the plane) is

$$D = \| \text{proj}_n \vec{PQ} \| = \left| \frac{\vec{PQ} \cdot \mathbf{n}}{\| \mathbf{n} \|} \right|$$

where $P$ is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$ ($a \neq 0$), let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point $(-d/a, 0, 0)$ lies in the plane.

**EXAMPLE 5  Finding the Distance Between a Point and a Plane**
\[ \text{Try It 5} \]

Q (1, 2, 3)  \[ n = \langle 2, -1, 1 \rangle \]

\[ 2x - y + z = 4 \]

\[ \overrightarrow{D} = \left| \overrightarrow{PQ} \cdot n \right| = \frac{\langle 1, 2, -1 \rangle \cdot \langle 2, -1, 1 \rangle}{\|n\|} = \frac{1 \cdot 2 - 2 \cdot -1 + 1 \cdot 1}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{6}} \]

Find \[ \overrightarrow{PQ} \]

2x - y + z = 4

p Any point on the plane

Q = (1, 2, 3)

Q = (2, 0, 0) or (0, 0, 0)

\[ \overrightarrow{PQ} = \langle 1, 2, -1 \rangle \]

Distance
the Geometry of Space

From Theorem 11.13, you can determine that the distance between the point \( Q(x_0, y_0, z_0) \) and the plane given by \( ax + by + cz + d = 0 \) is

\[
D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}
\]

or

\[
D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}
\]

where \( P(x_1, y_1, z_1) \) is a point in the plane and \( d = -(ax_1 + by_1 + cz_1) \).

**EXAMPLE 6 Finding the Distance Between Two Parallel Planes**

Find the distance between the two parallel planes given by

\[3x - y + 2z - 6 = 0\] and \[6x - 2y + 4z + 4 = 0.\]
Try It 6

\[ n = \langle 5, -3, 10 \rangle \]

\[ 5x - 3y + 10z = -20 \]

\[ (\frac{8}{3}, 0, 0), (9, -10, 0), (0, 0, \frac{3}{5}) \]

\[ P (l, l, 0) \]

Point In First Plane

Directions #1 of 2nd Plane

\[ \langle 5, -3, 10 \rangle \]

\[ D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \]

\[ D = \frac{57 + 13 + 20}{\sqrt{35 + 9 + 60}} = \frac{12}{\sqrt{134}} \approx 1.04 \text{ units} \]
The formula for the distance between a point and a line in space resembles the formula for the distance between a point and a plane—except that you replace the dot product of \( \mathbf{n} \) with the length of the cross product of \( \mathbf{n} \) and the normal vector \( \mathbf{u} \) with a direction vector for the line.

**THEOREM 11.14 Distance Between a Point and a Line in Space**

The distance between a point \( Q \) and a line in space is given by

\[
D = \frac{|\overrightarrow{PQ} \times \mathbf{u}|}{||\mathbf{u}||}
\]

where \( \mathbf{u} \) is a direction vector for the line and \( P \) is a point on the line.

**Proof** In Figure 11.54, let \( D \) be the distance between the point \( Q \) and the line. Then \( D = |\overrightarrow{PQ}| \sin \theta \), where \( \theta \) is the angle between \( \mathbf{u} \) and \( \overrightarrow{PQ} \). By Theorem 11.8, you have

\[
||\mathbf{u}|| |\overrightarrow{PQ}| \sin \theta = |\mathbf{u} \times \overrightarrow{PQ}| = |\overrightarrow{PQ} \times \mathbf{u}|.
\]

Consequently,

\[
D = |\overrightarrow{PQ}| \sin \theta = \frac{|\overrightarrow{PQ} \times \mathbf{u}|}{||\mathbf{u}||}.
\]
Try It 7

Distance Between \( Q (4, 1, -2) \)
\( u = \langle 2, 2, 1 \rangle \)

Line \( x = 2t + 2 \) \( y = 2t \) \( z = t - 3 \)

\( t = 0 \)

\( x = 2 \) \( y = 0 \) \( z = -3 \)

Point on Line \( P (2, 0, -3) \)

Direction \#s on Line \( \overrightarrow{PQ} = \langle 4 - 2, 1 - 0, -3 + 3 \rangle = \langle 2, 1, 1 \rangle \)

Magnitude

\( \| \langle 2, 1, 1 \rangle \times \langle 2, 1, 1 \rangle \| = \frac{\| -1i + 4j + 2k \|}{3} = \frac{\sqrt{9 + 16 + 4}}{3} = \frac{\sqrt{29}}{3} \)

\( \frac{\sqrt{9 + 4 + 4}}{3} = \frac{\sqrt{17}}{3} = \frac{\sqrt{17}}{3} \)
Cylindrical Surfaces

The first five sections of this chapter contained the vector portion of the preliminary work necessary to study vector calculus and the calculus of space. In this and the next section, you will study surfaces in space and alternative coordinate systems for space. You have already studied two special types of surfaces.

1. Spheres: \((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\)  
2. Planes: \(ax + by + cz + d = 0\)  

A third type of surface in space is called a **cylindrical surface**, or simply a cylinder. To define a cylinder, consider the familiar right circular cylinder shown in Figure 11.56. You can imagine that this cylinder is generated by a vertical line moving around the circle \(x^2 + y^2 = a^2\) in the xy-plane. This circle is called a **generating curve** for the cylinder, as indicated in the following definition.

**Definition of a Cylinder**

Let \(C\) be a curve in a plane and let \(L\) be a line not in a parallel plane. The set of all lines parallel to \(L\) and intersecting \(C\) is called a **cylinder**. \(C\) is called the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are ** rulings**.
A third type of surface in space is called a **cylindrical surface**, or simply a **cylinder**. To define a cylinder, consider the familiar right circular cylinder shown in Figure 11.56. You can imagine that this cylinder is generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the $xy$-plane. This circle is called a generating curve for the cylinder, as indicated in the following definition.

**Definition of a Cylinder**

Let $C$ be a curve in a plane and let $L$ be a line not in a parallel plane. The set of all lines parallel to $L$ and intersecting $C$ is called a **cylinder**. $C$ is called the generating curve (or **directrix**) of the cylinder, and the parallel lines are called rulings.

**NOTE** Without loss of generality, you can assume that $C$ lies in one of the three coordinate planes. Moreover, this text restricts the discussion to **right cylinders**—cylinders whose rulings are perpendicular to the coordinate plane containing $C$, as shown in Figure 11.57.

For the right circular cylinder shown in Figure 11.56, the equation of the generating curve is

$$x^2 + y^2 = a^2.$$
Cylindrical Surfaces

The first five sections of this chapter contained work necessary to study vector calculus and the section, you will study surfaces in space and alter.
You have already studied two special types of surface.
1. Spheres: $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$
2. Planes: $ax + by + cz + d = 0$

A third type of surface in space is called a cylinder. To define a cylinder, consider the familiar circle of radius $\alpha$ in the $xy$-plane. You can imagine that this cylinder is formed by translating the circle along the $z$-axis. Thus, the equation of a right circular cylinder is

$$x^2 + y^2 = \alpha^2$$

Right circular cylinder: $x^2 + y^2 = \alpha^2$

Rulings are parallel to z-axis.

**Figure 11.56**

Definition of a Cylinder

Let $C$ be a curve in a plane and let $L$ be a line.
Definition of a Cylinder

Let $C$ be a curve in a plane and let $L$ be a line of all lines parallel to $L$ and intersecting $C$ is a generating curve (or directrix) of the cylinder called rulings.

NOTE Without loss of generality, you can assume $L$ is a plane. Moreover, this text restricts the discussion to rulings that are perpendicular to the coordinate plane containing $C$.

For the right circular cylinder shown in Figure 11.57, the generating curve is

$$x^2 + y^2 = a^2.$$

To find an equation for the cylinder, note that $z$ is determined by fixing the values of $x$ and $y$ and then allowing $z$ to vary. Thus, the value of $z$ is arbitrary and is, therefore, a parameter. The equation of this cylinder is simply

$$x^2 + y^2 = a^2.$$
For the right circular cylinder shown in Figure 11.56, the equation of the generating curve is

$$x^2 + y^2 = a^2.$$  

Equation of generating curve in $xy$-plane

To find an equation for the cylinder, note that you can generate any one of the rulings by fixing the values of $x$ and $y$ and then allowing $z$ to take on all real values. In this sense, the value of $z$ is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2$$  

Equation of cylinder in space

**Equations of Cylinders**

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.
Try It One

\[ x^2 + z^2 = 16 \]

- \[ z = 0 \]
- \[ x = \pm 4 \]
- \[ z = 0 \]
- \[ x = \pm 4 \]

n. Hel \\
i. y-Axis \\
missing y component

Cylinder Running Parallel To y-Axis
parallel to the y-axis, as shown in Figure 11.58(b).

(a) Rulings are parallel to x-axis.

(b) Rulings are parallel to y-axis.
The fourth basic type of surface in space is a \textit{quadric surface}. Quadric surfaces are the three-dimensional analogs of conic sections.

\begin{quote}
\textbf{Quadric Surface}
\end{quote}

The equation of a \textit{quadric surface} in space is a second-degree equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$ 

There are six basic types of quadric surfaces: \textit{ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.}

The intersection of a surface with a plane is called the \textit{trace of the surface} in the plane. To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics. These traces, together with the \textit{standard form} of the equation of each quadric surface, are shown in the table on pages 812 and 813.
Sectors and the Geometry of Space

Ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

<table>
<thead>
<tr>
<th>Trace</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>Parallel to ( xy )-plane ( z = c )</td>
</tr>
<tr>
<td>Ellipse</td>
<td>Parallel to ( xz )-plane ( y = 0 )</td>
</tr>
<tr>
<td>Ellipse</td>
<td>Parallel to ( yz )-plane ( x = 0 )</td>
</tr>
</tbody>
</table>

The surface is a sphere if \( a = b = c \neq 0 \).

Hyperboloid of One Sheet

\[ x^2, \frac{y^2}{4}, z^2 \]
The surface is a sphere if 
\[ a = b = c \neq 0. \]

Hyperboloid of One Sheet

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

<table>
<thead>
<tr>
<th>Trace</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>Parallel to xy-plane ( z = 0 )</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>Parallel to xz-plane ( y = 0 )</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>Parallel to yz-plane ( x = 0 )</td>
</tr>
</tbody>
</table>

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.
### Hyperboloid of Two Sheets

\[ \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

<table>
<thead>
<tr>
<th>Trace</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>Parallel to xy-plane</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>Parallel to xz-plane</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>Parallel to yz-plane</td>
</tr>
</tbody>
</table>

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.
Elliptic Cone

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \]

Trace | Plane
--- | ---
Ellipse | Parallel to xy-plane
Hyperbola | Parallel to xz-plane
Hyperbola | Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.
Elliptic Paraboloid

\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Trace</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>Parallel to ( xy )-plane</td>
</tr>
<tr>
<td>Parabola</td>
<td>Parallel to ( xz )-plane</td>
</tr>
<tr>
<td>Parabola</td>
<td>Parallel to ( yz )-plane</td>
</tr>
</tbody>
</table>

The axis of the paraboloid corresponds to the variable raised to the first power.
Elliptic Paraboloid

\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Trace</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>Parallel to ( xy )-plane</td>
</tr>
<tr>
<td>Parabola</td>
<td>Parallel to ( xz )-plane</td>
</tr>
<tr>
<td>Parabola</td>
<td>Parallel to ( yz )-plane</td>
</tr>
</tbody>
</table>

The axis of the paraboloid corresponds to the variable raised to the first power.